CERN-PH-TH/2010-150

Magnetic field contribution to the last electron-photon scattering

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Abstract

When the cosmic microwave photons scatter electrons just prior to the decoupling of matter and radiation, magnetic fields do contribute to the Stokes matrix as well as to the scalar, vector and tensor components of the transport equations for the brightness perturbations. The magnetized electron-photon scattering is hereby discussed in general terms by including, for the first time, the contribution of magnetic fields with arbitrary direction and in the presence of the scalar, vector and tensor modes of the geometry. The propagation of relic vectors and relic gravitons is discussed for a varying magnetic field orientation and for different photon directions. The source terms of the transport equations in the presence of the relativistic fluctuations of the geometry are also explicitly averaged over the magnetic field orientations and the problem of a consistent account of the small-scale and large-scale magnetic field is briefly outlined.

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1 Formulation of the problem

The last electron-photon scattering is customarily discussed without the additional complication of a magnetic field. In numerical codes as well as in analytical estimates, the collisional contributions are evaluated as if electrons and ions were free right before last scattering[1]. The relativistic fluctuations of the geometry are included in the classic transport problem [1] either by using specific gauges [2] or with fully gauge-invariant methods. The resulting equations including both the source terms coming from electron-photon scattering and the relativistic fluctuations of the geometry form the set of transport equations which can be solved within various approaches either by truncating the system at a specific (maximal) multipole [2] or by using the integration along the line of sight [3]. One of the consequences of the consistent solution of the system of transport equations are estimates of the temperature and polarization inhomogeneities of the Cosmic Microwave Background (CMB in what follows). The recent WMAP 7 data [5, 6, 7, 8, 9, 10] are able to constrain the vanilla ΛCDM scenario (where Λ stands for the dark-energy component and CDM for the cold dark matter component). In the near future the ΛCDM scenario[2] will be tested not only in its minimal version but in its non-minimal extensions ranging from the addition of a stochastic background of relic tensor modes of the geometry to large-scale magnetic fields [11, 12].

In recent years there has been mounting evidence of the role played by magnetic fields at large scales [11, 12]. Why should magnetic fields be assumed in various processes ranging from star formation to cluster dynamics and completely neglected prior to last scattering? Why are magnetic fields overlooked in CMB physics while they are observed in galaxies clusters, superclusters and high-redshift quasars? There are no reasons for doing so unless one would implicitly assume that large-scale magnetism suddenly arose between hydrogen recombination and, say, the gravitational collapse of the protogalaxy. While it might well be that the latter situation is the one preferred by nature, it would be nice to have some direct empirical evidence less biased by speculations. To comply with the latter program, a specific approach has been tailored through the last few years [12] (see also [13]). The idea is, in a nutshell, to introduce consistently large-scale magnetic fields in all the steps leading to the estimate of CMB anisotropies and polarization. So far the program undertaken in [12] led to various results

- the large-scale magnetic fields have been included both at the level of the initial conditions as well as the level of the evolution equations for the standard adiabatic mode and for the other entropic initial conditions [13];
- the temperature and polarization anisotropies induced by the magnetized (adiabatic and entropic) initial conditions have been computed [14];
- the parameters of the magnetized background have been estimated (for the first time)

[2] Λ stands for the dark-energy component while CDM denotes the cold dark matter component.

in [15] by using the TT and TE correlations[3] measured by the WMAP collaboration.

There exist other approaches to the interplay between large-scale magnetic fields and CMB anisotropies (see [16, 17, 18, 19, 20] for an incomplete list of references; see [12] for a more thorough account of earlier results). The common characteristic of those approaches has been to neglect the scalar modes of the geometry and to focus the attention to the tensor and vector modes. The recent results [13, 14, 15] show, in contrast with previous guesses, that large scale magnetic fields alter the initial conditions and the dynamics of the scalar modes of the geometry. They consequently distort, in a computable manner, the temperature and polarization anisotropies.

A limitation common to nearly all studies on pre-decoupling magnetism has been so far the total absence of the effect of the magnetic field in the process of electron-photon scattering. In [13, 14, 15], for instance, the magnetic fields are included in the initial conditions and in all the relevant governing equations. The electron-photon scattering, however, is assumed to take place as if the magnetic fields were absent. The potential smallness of the effects does not justify its neglect since diverse small effects are often claimed to be detectable because of the purported control we now have on CMB foregrounds [21].

The consistent inclusion of magnetic fields in electron-photon scattering modifies qualitatively the standard lore since the geodesics of electrons and ions are be modified by the presence of the Lorentz force term in curved backgrounds. Absent the contribution of the magnetic field, the motion of electrons and ions depends only upon the incident electric field; but when the magnetic field is included the classic treatment (see, for instance [1]) must be adapted to the new situation.

The neglect of the role of the magnetic field in the electron-photon scattering has been recently relaxed in the guiding centre approximation [22, 23, 24, 25], and for a specified magnetic field orientation. The argument for keeping the direction fixed was essentially practical and in the present paper a general treatment will by developed along a twofold perspective

- the orientation of the magnetic field will be kept arbitrary so that the matrix elements either in in the Jones or in the Mueller calculus will depend not only upon the directions of the incident and of the outgoing radiation but also on the magnetic field orientation;

- after including the magnetic field in the scattering process the source terms for the transport equations of the scalar, vector and tensor modes of the geometry will be deduced explicitly.

The latter analysis is still lacking both in the present and in the earlier literature. The magnetized electron-photon scattering is often required in diverse astrophyiscal situations

[3] Following the standard shorthand terminology the TT correlations denote the temperature autocorrelations while the TE correlations denote the cross-correlation between the temperature and the E-mode polarization.

like in the physics of magnetized sun spots [28], or the theory of synchrotron emission [29, 30] whose results cannot be directly used since prior at last scattering electrons and ions are notoriously non-relativistic. Conversely some studies involving directly Thomson scattering in a magnetized environment [31] do not incorporate the fluctuations of the geometry and are also obtained using a preferential magnetic field orientation.

The layout of this paper is therefore the following. In section 2 the tenets of the Mueller and Jones calculus will be reviewed and the matrix elements for the magnetized electron-photon scattering presented. Section 3 introduces the scalar, vector and tensor components of the brightness perturbations and the calculation of the collisionless part of the transport equations. The full scalar, vector and tensor transport equations will be discussed, respectively, in sections 4, 5 and 6. Section 7 contains the concluding remarks. Explicit expressions involving all the relevant matrix elements both in the Jones and in the Mueller approaches have been collected in the appendices A and B.

2 Mueller and Jones calculus

In the Mueller calculus the Stokes parameters are organized in a four-dimensional (Mueller) column vector whose components are exactly the four Stokes parameters, i.e. I, Q, U and V. In the Jones calculus the electric fields of the wave are organized in a two-dimensional column vector and the Stokes parameters are effectively derived quantities (see [27] for an introduction to the Mueller and Jones approaches). Hereunder a hybrid approach shall be employed. The polarization tensor $\mathcal{P}_{ij} = \mathcal{P}_{ji} = E_i\, E_j^{*}$ can be organized in a Stokes matrix whose explicit form is:

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix} = \frac{1}{2}\left(I\,\mathbf{1} + U\,\sigma_1 + V\,\sigma_2 + Q\,\sigma_3\right), \tag{2.1}$$

where $\mathbf{1}$ denotes the identity matrix while σ_1, σ_2 and σ_3 are the three Pauli matrices. Sometimes the Stokes matrix $\mathcal{P}$ is separated in a traceless part (i.e. the polarization matrix) supplemented by the identity matrix multiplying the intensity of the radiation field: this separation shall not be employed here. The orientation of the coordinate system is illustrated in Fig. 1. The radial, azimuthal and polar directions are

$$\begin{aligned} \hat{r} &= (\cos\varphi\sin\vartheta,\, \sin\varphi\sin\vartheta,\, \cos\vartheta), \\ \hat{\vartheta} &= (\cos\varphi\cos\vartheta,\, \sin\varphi\cos\vartheta,\, -\sin\vartheta), \\ \hat{\varphi} &= (-\sin\varphi,\, \cos\varphi,\, 0), \end{aligned} \tag{2.2}$$

implying that $\hat{r}\times\hat{\vartheta} = \hat{\varphi}$. Photons propagate radially and $\hat{n} = (\vartheta, \varphi)$ denotes the direction of the scattered photon while $\hat{n}' = (\vartheta', \varphi')$ is the direction of the incoming photon; similarly $\mu = \cos\vartheta$ and $\nu = \cos\vartheta'$.

When the photons impinge the electrons in a magnetized environment the magnetic field can be treated in the guiding centre approximation. Denoting with $\vec{B}$ the comoving magnetic field intensity the guiding centre approximation [32, 33] stipulates

$$B_i(\vec{x}, \tau) \simeq B_i(\vec{x}_0, \tau) + (x^j - x_0^j)\partial_j B_i + ... \tag{2.3}$$

where the ellipses stand for the higher orders in the gradients leading, both, to curvature and drift corrections which will be neglected in this investigation. The scales one must therefore compare are $|\vec{x}_0| = L_0$, $|\vec{x} - \vec{x}_0| = L$, $\lambda_{\gamma}^{(\mathrm{rec})}$ (the wavelength of the incident radiation at the recombination epoch) and H_{rec}^{-1} (i.e. the Hubble rate at recombination). It is easy to appreciate that $\lambda_{\gamma}^{(\mathrm{rec})} = \mathcal{O}(\mu\mathrm{m})$ implying that

$$H_{\mathrm{rec}}^{-1} \simeq L \gg L_0 \gg \lambda_{\gamma}^{(\mathrm{rec})}. \tag{2.4}$$

Equation (2.4) implies that, for the purposes of the contribution of the magnetic field to the Stokes matrix the spatial gradients can be neglected while they cannot be neglected when estimating the effects of the large-scale inhomogeneities of the magnetic field. In spite of the fact that the contribution of the spatial gradients can be neglected in the first approximation, still the direction of the magnetic field should be appropriately taken into account. Consequently it is necessary to introduce a local basis which will define for us the magnetic field direction:

$$\begin{aligned}
\hat{e}_1 &= (\cos\alpha\cos\beta,\, \sin\alpha\cos\beta,\, -\sin\beta), \\
\hat{e}_2 &= (-\sin\alpha,\, \cos\alpha,\, 0), \\
\hat{e}_3 &= (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta).
\end{aligned} \tag{2.5}$$

The basis of Eq. (2.5) *local* since it accounts for the direction of the magnetic field over the typical scales involved in the electron-photon scattering. Once the direction of the local magnetic field has been fixed, the motion of the electrons and of the ions will follow the appropriate geodesics holding for charged particles in a gravitational field.

For the calculation of the scattering matrix the magnetic field can be aligned along $\hat{e}_3$. The latter choice is purely conventional and it does not prevent from varying arbitrarily the direction of the magnetic field with respect either to the direction of propagation of the photons or to the direction of propagation of the other fluctuations of the geometry. Consider, as an example, a situation which will be treated later on in greater detail, i.e. the case where a relic vector mode of the geometry [4] propagates along the direction $\hat{k}$. Since $\hat{k}$ cooincides also, by definition, with the direction of the Fourier wavevector the whole problem will be characterized by

- $(\hat{n} \cdot \hat{k})$, i.e. the projection of the photon momentum along the direction of propagation of the relic vector;

[4] The same discussion, with the due differences, can be repeated in the case of the scalar or tensor modes of the geometry. Here the case of the vector modes is just selected for sake of illustration.

- $(\hat{e}_3 \cdot \hat{k})$, i.e. the projection of the magnetic field direction along the direction of propagation of the relic vector.

In the mentioned example we can choose, without loss of generality, $\hat{k} = \hat{z}$ and the two physical polarizations of the relic vector will then be defined in the $\hat{x} - \hat{y}$ plane. In this situation $\cos\vartheta = \hat{k} \cdot \hat{n}$ and $\cos\alpha = \hat{k} \cdot \hat{e}_3$. The direction $\hat{e}_3$ does not coincide, in general, with $\hat{z}$. For instance if $\alpha = \beta = -\pi/2$, $\hat{e}_3$ coincides with $\hat{e}_y$ while for $\alpha = 0$ and $\beta = \pi/2$ $\hat{e}_3$ coincides with $\hat{e}_x$. This simple example shows explicitly that since the direction of $\hat{e}_3$ is

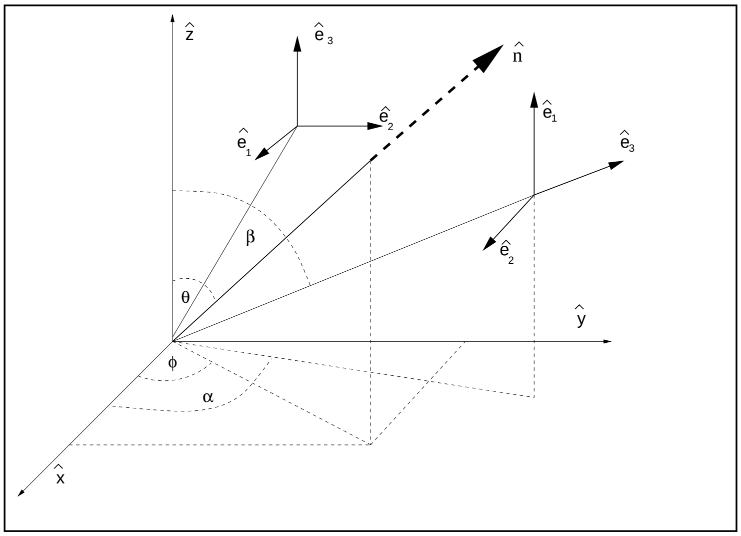

Figure 1: Schematic view of the relation between the coordinate system defining the scattered radiation field and the local frame of reference defining the direction of the magnetic field.

arbitrary, the orientation of the magnetic field is also generic. Such an arbitrariness entails the dependence of the scattering matrix upon two supplementary angles. In total the Stokes matrix will then depend upon the *two angles* defining the direction of the scattered radiation, the *two angles* defining the direction of the incident radiation and the *two angles* defining the direction of the magnetic field. The Stokes matrix will then depend overall upon six angles: (ϑ, φ) (for the directions of the scattered photons), (ϑ', φ') (for the directions of the incident photons) and (α, β) for the magnetic field direction. The schematic relation between the direction of the scattered radiation and the local frame defined by Eq. (2.5) is summarized in Fig. 1. The (thick) dashed line denotes the direction of $\hat{n}$, i.e. the direction of propagation of the radiation field. In Fig. 1 the two different alignments of $\hat{e}_3$ are just meant to illustrate the effective arbitrariness of the magnetic field orientation.

In the dipole approximation the scattered electric field can be computed as the composition of the scattered electric fields due to the electrons and to the ions:

$$\vec{E}^{\rm out}_{(\rm e)} = -e\frac{\vec{r} \times [\vec{r} \times \vec{a}_{(\rm e)}]}{r^3}, \qquad \vec{E}^{\rm out}_{(\rm i)} = e\frac{\vec{r} \times [\vec{r} \times \vec{a}_{(\rm i)}]}{r^3}, \tag{2.6}$$

where $\vec{a}_{(\mathrm{e})}$ and $\vec{a}_{(\mathrm{i})}$ are, respectively, the accelerations for the electrons and for the ions. In the local frame defined by Eq. (2.5) the vector $\vec{A} = (\vec{a}_{(\mathrm{e})} - \vec{a}_{(\mathrm{i})})$ the vector can be decomposed as $\vec{A} = (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3)$. Denoting with $E_1 = (\vec{E} \cdot \hat{e}_1)$, $E_2 = (\vec{E} \cdot \hat{e}_2)$ and $E_3 = (\vec{E} \cdot \hat{e}_2)$ the components of the electric fields of the incident radiation in the local frame we have that from the geodesics of electrons and ions

$$A_1 = \frac{\omega_{\mathrm{pe}}^2}{4\pi n_0} \zeta(\omega) \biggl[\Lambda_1 E_1 - i f_{\mathrm{e}} \Lambda_2 E_2\biggr], \tag{2.7}$$

$$A_2 = \frac{\omega_{\mathrm{pe}}^2}{4\pi n_0} \zeta(\omega) \biggl[\Lambda_1 E_2 + i f_{\mathrm{e}} \Lambda_2 E_1\biggr], \tag{2.8}$$

$$A_3 = -\frac{\omega_{\mathrm{pe}}^2}{4\pi n_0} \Lambda_3 E_3, \tag{2.9}$$

where because of the global neutrality of the plasma, $n_0 = \tilde{n}_0 a^3$ is the common comoving concentration of electrons and ions; $\omega_{\mathrm{Be,\,i}}$ and $\omega_{\mathrm{pe,\,i}}$ denote respectively the Larmor and plasma frequencies for electrons (and ions)

$$\omega_{\mathrm{Be,\,i}} = \frac{e \vec{B} \cdot \hat{e}_3}{m_{\mathrm{e,\,i}} a}, \qquad \omega_{\mathrm{pe,\,i}} = \sqrt{\frac{4\pi e^2 n_0}{m_{\mathrm{e,\,i}} a}}, \tag{2.10}$$

where $m_{\mathrm{e,\,i}}$ denote either the electron or the ion mass depending upon the relative subscript and $a(\tau)$ is the scale factor of a conformally flat geometry of Friedmann-Robertson-Walker type whose line element and metric tensor are defined as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = a^2(\tau)[d\tau^2 - d\vec{x}^2], \qquad g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}. \tag{2.11}$$

In Eqs. (2.7), (2.8) and (2.9) the functions Λ_i (with $i = 1, 2, 3$) as well as $\zeta(\omega)$ and all depend upon the angular frequency of the photon (i.e. $\omega = 2\pi\nu$) and are defined as:

$$\begin{aligned}
\Lambda_1(\omega) &= 1 + \left(\frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{pe}}^2}\right)\left(\frac{\omega^2 - \omega_{\mathrm{Be}}^2}{\omega^2 - \omega_{\mathrm{Bi}}^2}\right), \\
\Lambda_2(\omega) &= 1 - \left(\frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{pe}}^2}\right)\left(\frac{\omega_{\mathrm{Bi}}}{\omega_{\mathrm{Be}}}\right)\left(\frac{\omega^2 - \omega_{\mathrm{Be}}^2}{\omega^2 - \omega_{\mathrm{Bi}}^2}\right), \\
\Lambda_3(\omega) &= 1 + \left(\frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{pe}}^2}\right), \\
\zeta(\omega) &= \frac{\omega^2}{\omega_{\mathrm{Be}}^2 - \omega^2} = \frac{1}{f_{\mathrm{e}}^2(\omega) - 1}, \qquad f_{\mathrm{e}}(\omega) = \left(\frac{\omega_{\mathrm{Be}}}{\omega}\right).
\end{aligned} \tag{2.12}$$

The scale factor $a(\tau)$ appears explicitly in Eqs. (2.10) since the mass of the (non relativistic) species breaks the conformal invariance of the system of equations. Indeed, Eqs. (2.7), (2.8) and (2.9) follow from the geodesics of charged species in the conformally flat metric of Eq. (2.11) where, for a generic massive particle, the mass shell condition implies that

$g_{\alpha\beta}P^{\alpha}P^{\beta} = m^2$ ($P^{\alpha} = mu^{\alpha}$ is the canonical momentum and u^{α} the four-velocity). Recalling that the comoving three-momentum $\vec{q}$ is defined as $\vec{q} = a\vec{p}$ where $\delta_{ij}p^i p^j = -g_{ij}P^i P^j$, the comoving three-velocity is given by $\vec{v} = \vec{q}/\sqrt{q^2 + m^2 a^2}$. Since the electrons are non-relativistic at last scattering $\vec{q} = ma\vec{v}$ and this is, ultimately, the rationale for the appearance of the scale factors in the explicit expressions of the Larmor and plasma frequencies for the electrons and for the ions[5]. The numerical value of $f_{\rm e}(\omega)$ for typical cosmological parameters is given by

$$f_{\rm e}(\omega) = \left(\frac{\omega_{\rm Be}}{\omega}\right) = 2.79 \times 10^{-12}\left(\frac{B}{\rm nG}\right)\left(\frac{\rm GHz}{\nu}\right)(z_{*} + 1) \ll 1, \tag{2.13}$$

where z_{*} is the redshift to last scattering, i.e. $z_{*} = 1090.79^{+0.94}_{-0.92}$ according to the WMAP-7yr data [10]. In Eq. (2.13) $B = |\hat{e}_3 \cdot \vec{B}|$; grossly speaking the typical values of ν and B appearing in Eq. (2.13) do correspond, respectively, to the (very minimal) value of the frequency channel of CMB experiments and to the maximal value of the comoving magnetic field allowed by the distortions of the temperature autocorrelations and of the cross-correlations between temperature and polarization. The evolution of the Stokes matrix $\mathcal{P}$ can be formally written as[6]

$$\frac{d\mathcal{P}}{d\tau} + \epsilon'\mathcal{P} = \frac{3\epsilon'}{16\pi}\int M(\Omega,\Omega',\alpha,\beta)\,\mathcal{P}(\Omega,\Omega')\,M^{\dagger}(\Omega,\Omega',\alpha,\beta), \tag{2.14}$$

where $d\Omega' = d\cos\vartheta'\,d\varphi'$ and, defining the rate of electron-photon scattering $\Gamma_{\gamma{\rm e}}$,

$$\epsilon' = a\Gamma_{\gamma{\rm e}} = a\tilde{n}_0 x_{\rm e}\sigma_{{\rm e}\gamma}, \qquad \sigma_{\gamma{\rm e}} = \frac{8}{3}\pi r_{\rm e}^2, \qquad r_{\rm e} = \frac{e^2}{m_{\rm e}} \tag{2.15}$$

is the differential optical depth. At the right of Eq. (2.14) the matrix $M(\Omega,\Omega',\alpha,\beta)$ is a 2×2 and the four entries of the matrix $M(\Omega,\Omega',\alpha,\beta)$ are separately reported in appendix A. From the matrix elements of $M(\Omega,\Omega',\alpha,\beta)$ it is immediately possible to derive the evolution equations for the Stokes parameters in the Mueller form, namely,

$$\frac{d\mathcal{I}}{d\tau} + \epsilon'\mathcal{I} = \frac{3\epsilon'}{32\pi}\int d\Omega'\ \mathcal{T}(\Omega,\Omega',\alpha,\beta)\mathcal{I}(\Omega'), \tag{2.16}$$

where $\mathcal{I}$ is a column matrix whose entries are, respectively, I, Q, U and V. The entries of the 4×4 Mueller matrix will be denoted as $\mathcal{T}_{ij}(\Omega,\Omega',\alpha,\beta)$ where i and j run over the various Stokes parameters I, Q, U and V and are reported in appendix A (see, in particular, Eqs. (A.11)– (A.26)). Finally, in terms of the matrix the evolution equations of the different Stokes parameters can be formally written as

$$\frac{dI}{d\tau} + \epsilon' I = \frac{3\epsilon'}{32\pi}\int d\Omega'\mathcal{F}_I(\Omega,\Omega',\alpha,\beta), \tag{2.17}$$

$$\frac{dQ}{d\tau} + \epsilon' Q = \frac{3\epsilon'}{32\pi}\int d\Omega'\mathcal{F}_Q(\Omega,\Omega',\alpha,\beta), \tag{2.18}$$

[5]For more explicit discussions of these points see [15, 22]. Note that Eq. (A12) of [22] contains few trivial typos which have been corrected in the archive version of the same paper.

[6]The dagger in Eq. (2.14) defines, as usual, the complex conjugate of the transposed matrix.

$$\frac{dU}{d\tau} + \epsilon' U = \frac{3\epsilon'}{32\pi}\int d\Omega' \mathcal{F}_U(\Omega, \Omega', \alpha, \beta), \tag{2.19}$$
$$\frac{dV}{d\tau} + \epsilon' V = \frac{3\epsilon'}{32\pi}\int d\Omega' \mathcal{F}_V(\Omega, \Omega', \alpha, \beta), \tag{2.20}$$

where in all the integrands at the right hand side of Eqs. (2.17), (2.18), (2.19) and (2.20) the matrix elements $\mathcal{T}_{ij}(\Omega, \Omega', \alpha, \beta)$ are functions of, both, the angles of the incident radiation $\Omega' = (\vartheta', \varphi')$, the angles of the scattered radiation $\Omega = (\vartheta, \varphi)$ and the orientation of the magnetic field defined by the angles α and β. The explicit relations between $\mathcal{T}_{ij}(\Omega, \Omega', \alpha, \beta)$ and the matrix elements $M_{ij}(\Omega, \Omega', \alpha, \beta)$ are reported in the appendix A.

3 Brightness perturbations

The brightness perturbations, i.e. the fluctuations of the Stokes parameters in comparison to their equilibrium values can be decomposed as

$$\Delta_X(\vec{x}, \tau) = \Delta_X^{(\mathrm{s})}(\vec{x}, \tau) + \Delta_X^{(\mathrm{v})}(\vec{x}, \tau) + \Delta_X^{(\mathrm{t})}(\vec{x}, \tau), \tag{3.1}$$

where $X = I$, Q, U, V denotes, generically, one of the four Stokes parameters and where the superscripts refer, respectively, to the scalar, vector and tensor modes of the geometry. The scalar, vector and tensor components of the brightness perturbations are affected, respectively, by the scalar, vector and tensor inhomogeneties of the geometry and of the various sources. Assuming the conformally flat background introduced in Eq. (2.11), the fluctuations of the metric can be written, in general terms, as

$$\delta g_{\mu\nu}(\vec{x}, \tau) = \delta_{\mathrm{s}} g_{\mu\nu}(\vec{x}, \tau) + \delta_{\mathrm{v}} g_{\mu\nu}(\vec{x}, \tau) + \delta_{\mathrm{t}} g_{\mu\nu}(\vec{x}, \tau), \tag{3.2}$$

where δ_{s}, δ_{v} and δ_{t} denote the inhomogeneity preserving, separately, the scalar, vector and tensor nature of the fluctuations. The scalar modes of the geometry are parametrized in terms of four independent functions $\psi(\vec{x}, \tau)$, $\phi(\vec{x}, \tau)$, $E(\vec{x}, \tau)$ and $F(\vec{x}, \tau)$:

$$\begin{aligned}
\delta_{\mathrm{s}} g_{00}(\vec{x}, \tau) &= 2a^2(\tau)\phi(\vec{x}, \tau), \\
\delta_{\mathrm{s}} g_{0i}(\vec{x}, \tau) &= -a^2(\tau)\partial_i F(\vec{x}, \tau), \\
\delta_{\mathrm{s}} g_{ij}(\vec{x}, \tau) &= 2a^2(\tau)[\psi(\vec{x}, \tau)\delta_{ij} - \partial_i\partial_j E(\vec{x}, \tau)].
\end{aligned} \tag{3.3}$$

By setting E and F to zero the gauge freedom is completely fixed and this choice pins down the longitudinal (or conformally Newtonian) gauge. The vector modes are described by two independent vectors $Q_i(\vec{x}, \tau)$ and $W_i(\vec{x}, \tau)$

$$\delta_{\mathrm{v}} g_{0i}(\vec{x}, \tau) = -a^2 Q_i(\vec{x}, \tau), \qquad \delta_{\mathrm{v}} g_{ij}(\vec{x}, \tau) = a^2\Big[\partial_i W_j(\vec{x}, \tau) + \partial_j W_i(\vec{x}, \tau)\Big], \tag{3.4}$$

subjected to the conditions $\partial_i Q^i = 0$ and $\partial_i W^i = 0$. It will be convenient, for the present purposes, to choose the gauge $Q_i = 0$. The tensor modes of the geometry are parametrized

in terms of a rank-two tensor in three spatial dimensions, i.e.

$$\delta_t g_{ij}(\vec{x},\tau) = -a^2 h_{ij}, \qquad \partial_i h^i_j(\vec{x},\tau) = h^i_i(\vec{x},\tau) = 0, \tag{3.5}$$

which is automatically invariant under infinitesimal coordinate transformations. The following shorthand notation[7] will be adopted

$$\mathcal{L}^{(\mathrm{s})}_I(\hat{n},\vec{x},\tau) = \partial_\tau \Delta^{(\mathrm{s})}_I + \hat{n}^i\partial_i\Delta^{(\mathrm{s})}_I + \epsilon'\Delta^{(\mathrm{s})}_I + \frac{1}{q}\left(\frac{dq}{d\tau}\right)_{\mathrm{s}}, \tag{3.6}$$

$$\mathcal{L}^{(\mathrm{v})}_I(\hat{n},\vec{x},\tau) = \partial_\tau \Delta^{(\mathrm{v})}_I + \hat{n}^i\partial_i\Delta^{(\mathrm{v})}_I + \epsilon'\Delta^{(\mathrm{v})}_I + \frac{1}{q}\left(\frac{dq}{d\tau}\right)_{\mathrm{v}}, \tag{3.7}$$

$$\mathcal{L}^{(\mathrm{t})}_I(\hat{n},\vec{x},\tau) = \partial_\tau \Delta^{(\mathrm{t})}_I + \hat{n}^i\partial_i\Delta^{(\mathrm{t})}_I + \epsilon'\Delta^{(\mathrm{t})}_I + \frac{1}{q}\left(\frac{dq}{d\tau}\right)_{\mathrm{t}}, \tag{3.8}$$

where $q = \hat{n}_i q^i$ and where the scalar, vector and tensor contributions to the derivatives of the modulus of the comoving three-momentum are given, respectively, by

$$\left(\frac{dq}{d\tau}\right)_{\mathrm{s}} = -q\partial_\tau\psi + q\hat{n}^i\partial_i\phi, \tag{3.9}$$

$$\left(\frac{dq}{d\tau}\right)_{\mathrm{v}} = \frac{q}{2}\hat{n}^i\hat{n}^j(\partial_i\partial_\tau W_j + \partial_\tau\partial_j W_i), \tag{3.10}$$

$$\left(\frac{dq}{d\tau}\right)_{\mathrm{t}} = -\frac{q}{2}\,\hat{n}^i\,\hat{n}^j\,\partial_\tau h_{ij}. \tag{3.11}$$

The identities of Eqs. (3.9), (3.10) and (3.11) can be derived from the inhomogeneities of Eqs. (3.3)–(3.5) by recalling the definition of comoving three momentum (see discussion after Eq. (2.11)) and by using the relations

$$\frac{dx^i}{d\tau} = \frac{P^i}{P^0} = \frac{q^i}{q} = \hat{n}^i, \tag{3.12}$$

where P^i and P^0 are the space-like and time-like components of the canonical momentum obeying, for the photons, $g_{\alpha\beta}P^\alpha P^\beta = 0$. The notation introduced in Eqs. (3.6), (3.7), (3.8) for the fluctuations of the intensity can also be generalized to the linear and circular polarizations:

$$\mathcal{L}^{(\mathrm{y})}_X(\hat{n},\vec{x},\tau) = \partial_\tau \Delta^{(\mathrm{y})}_X + \hat{n}^i\partial_i\Delta^{(\mathrm{y})}_X + \epsilon'\Delta^{(\mathrm{y})}_X, \tag{3.13}$$

where the subscript can coincide, alternatively, with Q, U and V (i.e. $X = Q,\, U,\, V$) and the superscript denotes the transformation properties of the given fluctuation (i.e. $\mathrm{y} = \mathrm{s},\ \mathrm{v},\ \mathrm{t}$). The fluctuations of the geometry would seem to affect only the brightness perturbation for the intensity but such a conclusion would be incorrect: in the presence of a magnetic field the evolution equations of the four brightness perturbations are all coupled by the collision

[7]The partial derivations with respect to τ will be denotes by ∂_τ; the partial derivations with respect to the spatial coordinates will be instead denoted by ∂_i with $i = 1,\, 2,\, 3$.

term which does not only contain the intensity of the radiation field but a weighted sum of the four brightness perturbations integrated over the directions of the incident radiation. Consequently the polarization of the metric fluctuations will also impact on all the four brightness perturbations. The conventions on the Fourier transform and polarizations of the scalar, vector and tensor modes will be, in short,

$$\phi(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\phi(\vec{k},\tau)\,e^{i\vec{k}\cdot\vec{x}}, \tag{3.14}$$

$$\psi(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\psi(\vec{k},\tau)\,e^{i\vec{k}\cdot\vec{x}}, \tag{3.15}$$

$$W_i(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k W_i(\vec{k},\tau)\,e^{i\vec{k}\cdot\vec{x}}, \qquad \partial_i W^i(\vec{x},\tau) = 0, \tag{3.16}$$

$$h_{ij}(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k h_{ij}(\vec{k},\tau)\,e^{i\vec{k}\cdot\vec{x}}, \qquad \partial_i h^i_j(\vec{x},\tau) = h^i_i(\vec{x},\tau) = 0. \tag{3.17}$$

The vector and the tensor polarizations can be decomposed, respectively, as

$$W_i(\vec{k},\tau) = \sum_\lambda e_i^{(\lambda)} W_{(\lambda)}(\vec{k},\tau) = \hat{a}_i W_a(\vec{k},\tau) + \hat{b}_i W_b(\vec{k},\tau), \tag{3.18}$$

$$h_{ij}(\vec{k},\tau) = \sum_\lambda \epsilon_{ij}^{(\lambda)} h_{(\lambda)}(\vec{k},\tau) = \epsilon_{ij}^{\oplus} h_{\oplus}(\vec{k},\tau) + \epsilon_{ij}^{\otimes} h_{\otimes}(\vec{k},\tau), \tag{3.19}$$

where $\hat{k}$ denotes the direction of propagation and the two orthogonal directions $\hat{a}$ and $\hat{b}$ are such that $\hat{a}\times\hat{b} = \hat{k}$. Supposing that the direction of propagation of the relic tensor is oriented along $\hat{k}$, the two tensor polarizations are defined in terms of $\hat{a}_i$ and $\hat{b}_i$ as:

$$\epsilon_{ij}^{\oplus}(\hat{k}) = \hat{a}_i\hat{a}_j - \hat{b}_i\hat{b}_j, \qquad \epsilon_{ij}^{\otimes}(\hat{k}) = \hat{a}_i\hat{b}_j + \hat{a}_j\hat{b}_i. \tag{3.20}$$

The projections of the vector and of the tensor polarizations on the direction of photon propagation $\hat{n}$ are:

$$\hat{n}^i W_i(\vec{k},\tau) = \biggl[\hat{n}^i\hat{a}_i W_a(\vec{k},\tau) + \hat{n}^i\hat{b}_i W_b(\vec{k},\tau)\biggr], \tag{3.21}$$

$$\hat{n}^i\hat{n}^j h_{ij}(\vec{k},\tau) = \biggl\{[(\hat{n}\cdot\hat{a})^2 - (\hat{n}\cdot\hat{b})^2]h_{\oplus}(\vec{k},\tau) + 2(\hat{n}\cdot\hat{a})(\hat{n}\cdot\hat{b})h_{\otimes}(\vec{k},\tau)\biggr\}. \tag{3.22}$$

Choosing the direction of propagation of the relic vector and of the relic tensor along the $\hat{z}$ axis, the unit vectors $\hat{a}$ and $\hat{b}$ will coincide with the remaining two Cartesian directions and the related Fourier amplitudes will satisfy

$$\hat{n}^i W_i(\vec{k},\tau) = \sqrt{\frac{2\pi}{3}}\biggl[W_L(\vec{k},\tau)\,Y_1^{-1}(\vartheta,\varphi) - W_R(\vec{k},\tau)Y_1^{1}(\vartheta,\varphi)\biggr], \tag{3.23}$$

$$\hat{n}^i\hat{n}^j h_{ij}(\vec{k},\tau) = \biggl[h_R(\vec{k},\tau)Y_2^{2}(\vartheta,\varphi) + h_L(\vec{k},\tau)Y_2^{-2}(\vartheta,\varphi)\biggr], \tag{3.24}$$

where

$$W_L(\vec{k},\tau) = \frac{W_a(\vec{k},\tau) + iW_b(\vec{k},\tau)}{\sqrt{2}}, \qquad W_R(\vec{k},\tau) = \frac{W_a(\vec{k},\tau) - iW_b(\vec{k},\tau)}{\sqrt{2}},$$
$$h_L(\vec{k},\tau) = \frac{h_\oplus(\vec{k},\tau) + ih_\otimes(\vec{k},\tau)}{\sqrt{2}}, \qquad h_R(\vec{k},\tau) = \frac{h_\oplus(\vec{k},\tau) - ih_\otimes(\vec{k},\tau)}{\sqrt{2}}; \tag{3.25}$$

the spherical harmonics appearing in Eqs. (3.23) and (3.24) are, respectively,

$$Y_1^{\pm 1}(\vartheta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\,\sin\vartheta\; e^{\pm i\varphi}, \qquad Y_2^{\pm 2}(\vartheta,\varphi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\vartheta\; e^{\pm 2i\varphi}, \tag{3.26}$$

showing, as well known in the context of the total angular momentum method [34], that the vector and tensor modes excite, respectively, the two harmonics given in Eq. (3.26). While the total angular momentum method can be generalized to the case of an arbitrarily oriented magnetic field, we prefer to work, in the present context, with the formalism which is more directly applicable to numerical codes and to standard analytic estimates. In Fourier space Eqs. (3.6), (3.7) and (3.8) become

$$\mathcal{L}_I^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \partial_\tau\Delta_I^{(\mathrm{s})} + (ik\mu + \epsilon')\Delta_I^{(\mathrm{s})} + ik\mu\phi - \partial_\tau\psi, \tag{3.27}$$
$$\begin{aligned}\mathcal{L}_I^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) &= \partial_\tau\Delta_I^{(\mathrm{v})} + (ik\mu + \epsilon')\Delta_I^{(\mathrm{v})}\\ &+ \sqrt{\frac{2\pi}{3}}i\,\mu\left[\partial_\tau W_L(\vec{k},\tau)\,Y_1^{-1}(\vartheta,\varphi) - \partial_\tau W_R(\vec{k},\tau)Y_1^{1}(\vartheta,\varphi)\right],\end{aligned} \tag{3.28}$$
$$\begin{aligned}\mathcal{L}_I^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) &= \partial_\tau\Delta_I^{(\mathrm{t})} + (ik\mu + \epsilon')\Delta_I^{(\mathrm{t})}\\ &- \sqrt{\frac{2\pi}{15}}\left[\partial_\tau h_R(\vec{k},\tau)Y_2^{2}(\vartheta,\varphi) + \partial_\tau h_L(\vec{k},\tau)Y_2^{-2}(\vartheta,\varphi)\right].\end{aligned} \tag{3.29}$$

Similarly Eq. (3.13) becomes, in Fourier space,

$$\mathcal{L}_X^{(\mathrm{y})}(\mu,\varphi,\vec{k},\tau) = \partial_\tau\Delta_X^{(\mathrm{y})} + (ik\mu + \epsilon')\Delta_X^{(\mathrm{y})}. \tag{3.30}$$

The explicit form of the transport equations for the scalar, vector and tensor modes of the geometry will be scrutinized in the three forthcoming sections.

4 Scalar modes

Following the notation of Eqs. (3.27) and (3.30) the scalar transport equations can be formally expressed as

$$\mathcal{L}_I^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \epsilon'\hat{n}^i v_i^{(\mathrm{s})} + \frac{3\epsilon'}{32\pi}\int_{-1}^{1}d\nu\int_0^{2\pi}d\varphi'\mathcal{F}_I^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{4.1}$$
$$\mathcal{L}_Q^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1}d\nu\int_0^{2\pi}d\varphi'\mathcal{F}_{\mathrm{Q}}^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{4.2}$$

$$\mathcal{L}_U^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \; \mathcal{F}_{\mathrm{U}}^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{4.3}$$

$$\mathcal{L}_V^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \; \mathcal{F}_{\mathrm{V}}^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{4.4}$$

where $v_i^{(\mathrm{s})}$ denotes the scalar component of the baryon velocity field. In Eqs. (4.1)–(4.4) the source terms involve the integration over the incoming photon directions. Both the integration over φ' and ν can be performed explicitly and the final expressions are rather lengthy, as easily imaginable. To make the explicit equations more manageable without loosing any relevant information it is useful, in the following part of the present section, to write the results already in the physical limit, i.e. owing to the numerical values of the plasma and Larmor frequencies and recalling Eq. (2.12)

$$\begin{aligned} &\Lambda_1(\omega) = \Lambda_2(\omega) = \Lambda_3(\omega) = 1 + \mathcal{O}(m_{\mathrm{e}}/m_{\mathrm{p}}), \\ &\zeta(\omega) \simeq -1 + f_{\mathrm{e}}^2(\omega) + \mathcal{O}(f_{\mathrm{e}}^4). \end{aligned} \tag{4.5}$$

The scalar source terms depend upon the explicit form of the matrix elements appearing in Eqs. (A.7), (A.8), (A.9) and (A.10). The integration over φ' can be performed explicitly. Using the notation

$$\overline{\mathcal{T}}_{ab}(\mu,\nu,\varphi,\alpha,\beta) = \int_0^{2\pi} d\varphi' \, \mathcal{T}_{ab}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{4.6}$$

the final results are reported, for completeness and future peruse, in appendix B. According to Eqs. (4.1)–(4.4), the expressions reported in Eqs. (B.1)-(B.16) must be integrated over ν. For the ν integration it is useful to expand the various brightness perturbations in a series of Legendre polynomials $P_\ell(\nu)$

$$\Delta_X(\nu,k,\tau) = \sum_\ell (-i)^\ell (2\ell+1)\, P_\ell(\nu)\, \Delta_{X\,\ell}(k,\tau). \tag{4.7}$$

The integration over ν will then have the net result of expressing the source terms in terms of a limited number of multipoles of the intensity and of the polarization. In explicit terms the source terms can be expressed, for each brightness perturbation, as an expansion in $f_{\mathrm{e}}(\omega)$:

$$\partial_\tau \Delta_I^{(\mathrm{s})} + (ik\mu + \epsilon')\Delta_I^{(\mathrm{s})} = \partial_\tau\psi - ik\mu\phi + \epsilon'\mathcal{A}_I + \epsilon' \, f_{\mathrm{e}}(\omega)\,\mathcal{B}_I + \epsilon' \, f_{\mathrm{e}}^2(\omega)\,\mathcal{C}_I, \tag{4.8}$$

$$\partial_\tau \Delta_Q^{(\mathrm{s})} + (ik\mu + \epsilon')\Delta_Q^{(\mathrm{s})} = \epsilon'\mathcal{A}_Q + \epsilon' \, f_{\mathrm{e}}(\omega)\,\mathcal{B}_Q + \epsilon' \, f_{\mathrm{e}}^2(\omega)\,\mathcal{C}_Q, \tag{4.9}$$

$$\partial_\tau \Delta_U^{(\mathrm{s})} + (ik\mu + \epsilon')\Delta_U^{(\mathrm{s})} = \epsilon'\mathcal{A}_U + \epsilon' \, f_{\mathrm{e}}(\omega)\,\mathcal{B}_U + \epsilon' \, f_{\mathrm{e}}^2(\omega)\,\mathcal{C}_U, \tag{4.10}$$

$$\partial_\tau \Delta_V^{(\mathrm{s})} + (ik\mu + \epsilon')\Delta_V^{(\mathrm{s})} = \epsilon'\mathcal{A}_V + \epsilon' \, f_{\mathrm{e}}(\omega)\,\mathcal{B}_V + \epsilon' \, f_{\mathrm{e}}^2(\omega)\,\mathcal{C}_V, \tag{4.11}$$

where, for $X = I$, Q, U, V, $\mathcal{A}_X$ denotes the leading order result, $\mathcal{B}_X$ denotes the next-to-leading order (NLO) correction while $\mathcal{C}_X$ denotes the next-to-next-to-leading (NNLO) term. Defining with S_P the usual combination of the quadrupole of the intensity and of the monopole and quadrupole of the linear polarization (see, e.g. [2, 3, 4])

$$S_P = \Delta_{I2} + \Delta_{Q0} + \Delta_{Q2}, \tag{4.12}$$

the leading order contribution for the for brightness perturbations is then given by:

$$\mathcal{A}_I = \Delta_{I0} + \mu v_{\rm b} - \frac{P_2(\mu)}{2} S_P, \tag{4.13}$$

$$\mathcal{A}_Q = \frac{3}{4}(1-\mu^2) S_P, \tag{4.14}$$

$$\mathcal{A}_U = 0, \qquad \mathcal{A}_V = -\frac{3}{2}\, i\, \mu\, \Delta_{V1}, \tag{4.15}$$

where the notation $\vec{v}^{(\rm s)} = \vec{k} v_{\rm b}$ has been employed for the scalar component of the Doppler term. The NLO contribution to the right hand side of Eqs. (4.8), (4.9), (4.10) and (4.11) is

$$\mathcal{B}_I = -\frac{3}{2}\, i \left[(1+\mu^2)\cos\beta + \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\right]\Delta_{V1} \tag{4.16}$$

$$\mathcal{B}_Q = -\frac{3}{2}\, i\, \Delta_{V1}\left[(\mu^2-1)\cos\beta + \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\right], \tag{4.17}$$

$$\mathcal{B}_U = \frac{3}{2}\, i\, \Delta_{V1}\sqrt{1-\mu^2}\sin\beta\sin(\varphi-\alpha), \tag{4.18}$$

$$\begin{aligned}\mathcal{B}_V &= \left[\mu\cos\beta - \frac{\sqrt{1-\mu^2}}{2}\cos(\varphi-\alpha)\sin\beta\right]\Delta_{I0}\\ &- \left[2\mu\cos(\varphi-\alpha)\sin^2\beta + \frac{\sqrt{1-\mu^2}}{2}\sin\beta\right]\Delta_{I2}\\ &- \left[\frac{\mu}{2}\cos\beta - \frac{\sqrt{1-\mu^2}}{4}\cos(\varphi-\alpha)\sin\beta\right](\Delta_{Q2}+\Delta_{Q0}).\end{aligned} \tag{4.19}$$

Finally, the NNLO contribution to the right hand side of Eqs. (4.8), (4.9), (4.10) and (4.11) is

$$\begin{aligned}\mathcal{C}_I &= \left[(\mu^2+1) + \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin 2\beta\right]\left(\frac{\Delta_{I0}}{2} - \Delta_{I2}\right)\\ &+ \frac{1}{2}\left[\mu^2(\Delta_{I0}+\Delta_{I2}) + (\Delta_{I2} - 2\Delta_{I0})\right]\sin^2\beta\cos^2(\varphi-\alpha) - \frac{\cos 2\beta}{2}(1+3\mu^2)\\ &- \left\{(1-\mu^2)\sin^2(\varphi-\alpha) + \frac{1+\mu^2}{4}\left[\cos 2(\varphi-\alpha-\beta) + \cos 2(\varphi-\alpha+\beta)\right]\right.\\ &- \left.\frac{\mu\sqrt{1-\mu^2}}{2}\left[\sin(\varphi-\alpha-2\beta) - \sin(\varphi-\alpha+2\beta)\right]\right\}(\Delta_{Q0}+\Delta_{Q2}),\end{aligned} \tag{4.20}$$

$$\begin{aligned}\mathcal{C}_Q &= \left\{\frac{\mu^2-1}{2} + \frac{\sin^2\beta}{8}\left[4 - 2(2\mu^2+1)\cos^2(\varphi-\alpha)\right] + \frac{\mu\sqrt{1-\mu^2}}{2}\cos(\varphi-\alpha)\sin 2\beta\right\}\Delta_{I0}\\ &- \left\{\frac{\mu^2-1}{4} + \frac{\sin^2\beta}{2}\left[1 + (1-\mu^2)\cos^2(\varphi-\alpha)\right] + \frac{\mu\sqrt{1-\mu^2}}{4}\cos(\varphi-\alpha)\sin 2\beta\right\}\Delta_{I2}\\ &+ \left\{\frac{\mu^2-1}{8} - \frac{\mu^2+1}{8}\cos 2(\varphi-\alpha) + \frac{3}{8}(1-\mu^2)\cos 2\beta\right.\\ &+ \frac{\mu^2+1}{16}\left[\cos 2(\varphi-\alpha-\beta) + \cos 2(\varphi-\alpha+\beta)\right]\\ &+ \left.\frac{\mu\sqrt{1-\mu^2}}{8}\left[\sin(\varphi-\alpha-2\beta) - \sin(\varphi-\alpha+2\beta)\right]\right\}(\Delta_{Q0}+\Delta_{Q2}),\end{aligned} \tag{4.21}$$

$$
\begin{aligned}
\mathcal{C}_U &= \sin(\varphi-\alpha)\Big[\frac{\sqrt{1-\mu^2}}{2}\sin 2\beta - \mu\cos(\varphi-\alpha)\sin^2\beta\Big]\Delta_{I0} \\
&+ \sin(\varphi-\alpha)\Big[\mu\cos(\varphi-\alpha)\sin^2\beta - \frac{\sqrt{1-\mu^2}}{4}\sin 2\beta\Big]\Delta_{I2} \\
&+ \frac{\sin\beta\sin(\varphi-\alpha)}{2}\Big[\sqrt{1-\mu^2}\cos\beta + 2\mu\cos(\varphi-\alpha)\sin\beta\Big](\Delta_{Q2}+\Delta_{Q0}), \qquad (4.22)\\
\mathcal{C}_V &= -\frac{3}{2}\,i\,\Big[\mu\cos^2\beta + \sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\cos\beta\Big]\Delta_{V1}. \qquad (4.23)
\end{aligned}
$$

Several cross-checks on the obtained results have been made; they will be swiftly mentioned and can be directly reproduced by using the results reported in the appendices A and B:

- it has been verified explicitly at the level of the exact expressions (i.e. without implementing the limit of Eq. (4.5)) the equations must be independent upon the α and β once $(\hat{e}_3 \cdot \vec{B}) \to 0$: this is exactly what happens;
- it has been verified that in the limit $\alpha = \beta = 0$ the exact expressions must reproduce the partial results already obtained in [23, 25, 26]; with Eqs. (4.8)–(4.11) few typos present in the published version of [23] are corrected;
- by averaging of the source terms over α and β terms proportional to $f_{\mathrm{e}}(\omega)$ should automatically disappear without performing any specific limit: this is what will be explicitly shown in the remaining part of this section.

The remaining part of the section is devoted to the averaging of the source terms over the magnetic field directions as suggested in the last point of the above list of items. By integrating over α and β the source functions appearing at the right hand side of Eqs. (4.1), (4.2), (4.3) and (4.4), the evolution equations for the brightness perturbations read

$$
\begin{aligned}
&\mathcal{L}_I^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \epsilon'\hat{n}^i v_i^{(\mathrm{s})} \\
&+\frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu\int_0^{2\pi} d\varphi'\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi} d\alpha\mathcal{F}_I^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \qquad (4.24)\\
&\mathcal{L}_Q^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu\int_0^{2\pi} d\varphi'\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi} d\alpha\mathcal{F}_{\mathrm{Q}}^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \qquad (4.25)\\
&\mathcal{L}_U^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu\int_0^{2\pi} d\varphi'\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi} d\alpha\mathcal{F}_{\mathrm{U}}^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \qquad (4.26)\\
&\mathcal{L}_V^{(\mathrm{s})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu\int_0^{2\pi} d\varphi'\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi} d\alpha\mathcal{F}_{\mathrm{V}}^{(\mathrm{s})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \qquad (4.27)
\end{aligned}
$$

where the factor $128\pi^2$ accounts for the 4π factor arising from the average over the solid angle spanned by α and β. By performing the averages explicitly, the evolution equations of the four brightness perturbations read:

$$
\partial_\tau\Delta_I^{(\mathrm{s})} + (ik\mu+\epsilon')\Delta_I^{(\mathrm{s})} = \partial_\tau\psi - ik\mu\phi + \epsilon'\Big[\Delta_{I0} + \mu v_{\mathrm{b}} - \frac{P_2(\mu)}{2}S_{\mathrm{P}}
$$

$$
\begin{aligned}
&+ \quad f_{\rm e}^2\biggl(\frac{2}{3}\Delta_{I0} + \frac{P_2(\mu)}{6} S_{\rm P}\biggr)\biggr] && (4.28)\\
\partial_{\tau}\Delta_{Q}^{(\rm s)} + (ik\mu + \epsilon')\Delta_{Q}^{(\rm s)} &= \epsilon' \frac{(f_{\rm e}^2 - 3)(\mu^2 -1)}{4} S_{\rm P}, && (4.29)\\
\partial_{\tau}\Delta_{U}^{(\rm s)} + (ik\mu + \epsilon')\Delta_{U}^{(\rm s)} &= 0, && (4.30)\\
\partial_{\tau}\Delta_{V}^{(\rm s)} + (ik\mu + \epsilon')\Delta_{V}^{(\rm s)} &= -\frac{i\,\epsilon'}{2}(3 + f_{\rm e}^2)\Delta_{V1}. && (4.31)
\end{aligned}
$$

The results of the present section make quantitatively clear that if the magnetic field has a predominant direction over typical scales comparable with the wavelengths of the scattered photons, then the circular polarization is larger than in the case where, over the same physical scales the magnetic field is randomly oriented. It has been argued in [22, 23] (see also [26]) that over small angular scales the maximal amount of circular polarization arises when there is a strong alignment of the magnetic field along the direction of propagation of the photon which coincides, for large multipoles, with the third Cartesian direction.

The present results improve and confirm, at once, the assumptions made in the analytic and numerical estimates of magnetized CMB anisotropies of Refs. [13, 14, 15]. Indeed, the scalar fluctuations of the geometry obey a set of evolution equations where large-scale magnetic fields contribute in many respects. These equations will not be repeated here and can be found in [13, 14, 15]. The present results improve on the transport equations used there and pave the way for a more consistent account the effects of pre-decoupling magnetic fields both at large as well as at small angular scales.

5 Vector modes

In the case of the vector modes of the geometry the integration over φ' of the source functions cannot be easily performed as in the case of the scalar modes of the geometry (see section 4). Each of the two vector polarizations induce a different angular dependence in the corresponding brightness perturbations. In this paper three categories of circular and linear polarizations can be defined:

- the linear and circular polarizations of the scattered (and incident) photons (already described in sections 2 and 3) which are described by the four Stokes parameters or by the appropriate Stokes matrix;

- the linear and circular polarizations of the relic vector waves which we are going to discuss in the present section and which have been already introduced, respectively, in Eqs. (3.21) and (3.23);

- the linear and circular polarizations of the relic tensor waves introduced, respectively, in Eqs. (3.22) and (3.24) and discussed in the following section 4.

The linear and circular polarizations of the relic tensor and vector waves are just equivalent basis for the description of the tensor and vector modes of the geometry. To avoid potential confusions the vector and the tensor waves will always be treated in the basis of the linear polarizations. In full analogy with the treatment of section 4 the evolution equations for the vector components of the brightness perturbations can be formally written as

$$\mathcal{L}_{I}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \epsilon' n^{i} v_{i}^{(\mathrm{v})} + \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{I}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.1}$$

$$\mathcal{L}_{Q}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{\mathrm{Q}}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.2}$$

$$\mathcal{L}_{U}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{\mathrm{U}}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.3}$$

$$\mathcal{L}_{V}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{\mathrm{V}}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.4}$$

where $v_{i}^{(\mathrm{v})}$ denotes the vector component of the baryon velocity field. The polarizations of the baryon velocity will follow the same kind of decomposition illustrated for the vector of the geometry in Eq. (3.23). The relative directions of the magnetic field intensity and of the photon propagation determine the polarization of the outgoing radiation. Following the strategy described in section 2 the direction of propagation of the relic vector wave can be fixed and the direction of the magnetic field varied at wish.

Consider first the case where the magnetic field is oriented along the same direction of the vector wave and suppose, without loss of generality, that the vector propagates along $\hat{k} = \hat{z}$. Since, in this case, $\alpha = \beta = 0$ and Eqs. (A.1), (A.2), (A.3) and (A.4) will lead, respectively, to the following matrix elements:

$$\begin{aligned}
M_{11}(\mu,\varphi,\nu,\varphi') &= \zeta\mu\nu\Lambda_1 \cos(\varphi'-\varphi) - \sqrt{1-\mu^2}\sqrt{1-\nu^2}\Lambda_3 \\
&- i\Lambda_2 f_{\mathrm{e}}\zeta\mu\nu \sin(\varphi'-\varphi), \\
M_{12}(\mu,\varphi,\nu,\varphi') &= -\zeta\mu\Lambda_1 \sin\Delta\varphi - i\Lambda_2 f_{\mathrm{e}}\zeta\mu\cos(\varphi'-\varphi), \\
M_{21}(\mu,\varphi,\nu,\varphi') &= \zeta\nu\Lambda_1 \sin\Delta\varphi + i f_{\mathrm{e}}\Lambda_2\zeta\nu\cos(\varphi'-\varphi), \\
M_{22}(\mu,\varphi,\nu,\varphi') &= \zeta\Lambda_1\cos(\varphi'-\varphi) - i f_{\mathrm{e}}\Lambda_2\zeta\sin(\varphi'-\varphi).
\end{aligned} \tag{5.5}$$

Using Eq. (5.5) the source terms of Eqs. (5.1), (5.2), (5.3) and (5.4) can be computed. The explicit form of Eqs. (5.1)–(5.4) is rather lengthy: instead of writing *all* the equations, Eq. (5.1) will just be written for illustration with the purpose of demonstrating how the different vector polarizations induce a specific azimuthal dependence in the vector brightness perturbations. Equation (5.1) written in the basis of the linear polarizations reads

$$\begin{aligned}
&\partial_{\tau}\Delta_{I}^{(\mathrm{v})} + (ik\mu + \epsilon')\Delta_{I}^{(\mathrm{v})} + i\,\mu\sqrt{1-\mu^2}\Big[\cos\varphi\partial_{\tau}W_a + \sin\varphi\partial_{\tau}W_b\Big] \\
&= \epsilon'\sqrt{1-\mu^2}[\cos\varphi v_a + \sin\varphi v_b] + \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu\int_{0}^{2\pi} d\varphi'\,\mathcal{F}_{I}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta),
\end{aligned} \tag{5.6}$$

where, according to Eq. (A.7) the integrand of the source term acts on the vector components of the various brightness perturbations and it is given by

$$\mathcal{F}_I^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta) = \mathcal{T}_{II}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_I^{(\mathrm{v})}(\nu,\varphi') + \mathcal{T}_{IQ}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_Q^{(\mathrm{v})}(\nu,\varphi')$$
$$+\mathcal{T}_{IU}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_U^{(\mathrm{v})}(\nu,\varphi') + \mathcal{T}_{IV}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_V^{(\mathrm{v})}(\nu,\varphi'). \quad (5.7)$$

After inspection of all the four expressions appearing in Eqs. (5.1), (5.2), (5.3) and (5.4) it can be checked that the consistent ansatz for the four brightness perturbations is given by

$$\Delta_I^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \sqrt{1-\mu^2}\biggl[\cos\varphi\mathcal{M}_a(k,\tau) + \sin\varphi\mathcal{M}_b(k,\tau)\biggr], \quad (5.8)$$

$$\Delta_Q^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \mu\sqrt{1-\mu^2}\biggl[\cos\varphi\mathcal{N}_a(k,\tau) + \sin\varphi\mathcal{N}_b(k,\tau)\biggr], \quad (5.9)$$

$$\Delta_U^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \sqrt{1-\mu^2}\biggl[-\sin\varphi\mathcal{N}_a(k,\tau) + \cos\varphi\mathcal{N}_b(k,\tau)\biggr], \quad (5.10)$$

$$\Delta_V^{(\mathrm{v})}(\varphi,\mu,\tau) = \sqrt{1-\mu^2}\biggl[\cos\varphi\mathcal{V}_a(k,\tau) + \sin\varphi\mathcal{V}_b(k,\tau)\biggr]. \quad (5.11)$$

The equations obeyed by $\mathcal{M}_a$, $\mathcal{N}_a$ and $\mathcal{V}_a$ are the same as the ones obeyed by $\mathcal{M}_b$, $\mathcal{N}_b$ and $\mathcal{V}_b$ and they can be written, for a generic linear polarization, as

$$\partial_\tau\mathcal{M} + (ik\mu+\epsilon')\mathcal{M} + i\mu\partial_\tau W = \epsilon' v + \epsilon'\mu\zeta\Lambda_1\Lambda_3\Sigma_1^{(\mathrm{v})} + \epsilon'\mu f_{\mathrm{e}}\zeta\Lambda_2\Lambda_3\Sigma_2^{(\mathrm{v})}, \quad (5.12)$$
$$\partial_\tau\mathcal{N} + (ik\mu+\epsilon')\mathcal{N} = \epsilon'\zeta\Lambda_1\Lambda_3\Sigma_1^{(\mathrm{v})} + \epsilon' f_{\mathrm{e}}\Lambda_2\Lambda_3\Sigma_2^{(\mathrm{v})}, \quad (5.13)$$
$$\partial_\tau\mathcal{V} + (ik\mu+\epsilon')\mathcal{V} = \epsilon' f_{\mathrm{e}}\zeta\Lambda_2\Lambda_3\Sigma^{(\mathrm{v})} + \epsilon'\zeta\Lambda_1\Lambda_3\Sigma_2^{(\mathrm{v})}, \quad (5.14)$$

where the two newly defined source functions $\Sigma_1^{(\mathrm{v})}$ and $\Sigma_2^{(\mathrm{v})}$ are given by:

$$\Sigma_1^{(\mathrm{v})}(k,\tau) = \frac{3}{8}\int_{-1}^{1}\biggl[\nu(\nu^2-1)\mathcal{M}(k,\nu,\tau) + (\nu^4-1)\mathcal{N}(k,\nu,\tau)\biggr]\,d\nu, \quad (5.15)$$
$$\Sigma_2^{(\mathrm{v})}(k,\tau) = \frac{3}{8}\int_{-1}^{1}(\nu^2-1)\,\mathcal{V}(k,\nu,\tau)\,d\nu. \quad (5.16)$$

The source functions appearing in Eqs. (5.15) and (5.16) can be made more explicit by expanding $\mathcal{M}(\nu,k,\tau)$, $\mathcal{N}(\nu,k,\tau)$ and $\mathcal{V}(\nu,k,\tau)$ with the same conventions employed in Eq. (4.7):

$$\mathcal{M}(\nu,k,\tau) = \sum_\ell(-i)^\ell(2\ell+1)\,P_\ell(\nu)\,\mathcal{M}_\ell(k,\tau). \quad (5.17)$$
$$\mathcal{N}(\nu,k,\tau) = \sum_\ell(-i)^\ell(2\ell+1)\,P_\ell(\nu)\,\mathcal{N}_\ell(k,\tau). \quad (5.18)$$
$$\mathcal{V}(\nu,k,\tau) = \sum_\ell(-i)^\ell(2\ell+1)\,P_\ell(\nu)\,\mathcal{V}_\ell(k,\tau), \quad (5.19)$$

where, following the same conventions of Eq. (4.7), $\mathcal{M}_\ell$, $\mathcal{N}_\ell$ and $\mathcal{V}_\ell$ denote the ℓ-th multipole of the corresponding quantity. The result of the integration over ν is therefore

$$\Sigma_1^{(\mathrm{t})}(k,\tau) = \frac{6}{35}\mathcal{N}_4 - \frac{3}{7}\mathcal{N}_2 - \frac{\mathcal{N}_0}{6} + \frac{3}{10}\,i\,(\mathcal{M}_1+\mathcal{M}_3), \quad (5.20)$$
$$\Sigma_2^{(\mathrm{t})}(k,\tau) = -\frac{\mathcal{V}_2}{2} - \frac{\mathcal{V}_4}{4}, \quad (5.21)$$

The same considerations developed in the basis of the linear vector polarizations can be repeated in the case of the left and right polarized waves. Bearing in mind Eq. (3.26), Eqs. (5.8)–(5.11) can be written:

$$\Delta_{I}^{(\mathrm{v})}(\varphi,\mu,k,\tau) = 2\sqrt{\frac{\pi}{3}}\Big[Y_{1}^{-1}(\mu,\varphi)\mathcal{M}_{L}(k,\tau) - Y_{1}^{-1}(\mu,\varphi)\mathcal{M}_{R}(k,\tau)\Big], \tag{5.22}$$

$$\Delta_{Q}^{(\mathrm{v})}(\varphi,\mu,k,\tau) = 2\mu\sqrt{\frac{\pi}{3}}\Big[Y_{1}^{-1}(\mu,\varphi,k,\tau)\mathcal{N}_{L}(k,\tau) - Y_{1}^{-1}(\mu,\varphi)\mathcal{N}_{R}(k,\tau)\Big], \tag{5.23}$$

$$\Delta_{U}^{(\mathrm{v})}(\varphi,\mu,k,\tau) = -2i\sqrt{\frac{\pi}{3}}\Big[Y_{1}^{-1}(\mu,\varphi)\mathcal{N}_{L}(k,\tau) + Y_{1}^{-1}(\mu,\varphi,k,\tau)\mathcal{N}_{R}(k,\tau)\Big], \tag{5.24}$$

$$\Delta_{V}^{(\mathrm{v})}(\varphi,\mu,k,\tau) = 2\sqrt{\frac{\pi}{3}}\Big[Y_{1}^{-1}(\mu,\varphi)\mathcal{V}_{L}(k,\tau) - Y_{1}^{-1}(\mu,\varphi)\mathcal{V}_{R}(k,\tau)\Big]. \tag{5.25}$$

For sufficiently small angular scales (i.e. for sufficiently large multipoles) the microwave sky degenerates into a plane and the. In this situation microwave photons propagate, for all practical purposes, along the $\hat{z}$ axis and instead of the spherical decomposition based on spherical harmonics one can safely use a plane-wave decomposition. Since the wavelength of the photons is typically much shorter than the inhomogeneity scale of the magnetic field one could also argue, at this point, that the situation in which the magnetic field oriented along the direction of propagation of the relic (vector) wave is sufficiently generic. This conclusion should however be scrutinized more carefully and this is the purpose of the discussion reported hereunder.

Suppose that the direction of propagation of the vector wave is not parallel to the magnetic field direction but orthogonal. If the relic vector propagates along the magnetic field direction, then, in the language of Eq. (2.5), $\hat{k} \parallel \hat{e}_3$ implying $\alpha = \beta = 0$. If the relic vector propagates orthogonally to the magnetic field direction then we can set $\alpha = \beta = -\pi/2$ implying that $\hat{k} \perp \hat{e}_3$. The direction of $\hat{k}$ will still be chosen to be the $\hat{z}$ axis so that $(\hat{k}\cdot\hat{n}) = \mu = \cos\vartheta$. In the case $\alpha = \beta = -\pi/2$ Eqs. (A.1), (A.2), (A.3) and (A.4) read

$$\begin{aligned}
M_{11}(\mu,\varphi,\nu,\varphi') &= \zeta\Lambda_1\sqrt{1-\mu^2}\sqrt{1-\nu^2} + \zeta\Lambda_1\mu\nu\cos\varphi'\cos\varphi - \Lambda_3\mu\nu\sin\varphi\sin\varphi' \\
&+ i f_{\mathrm{e}}\Lambda_2\zeta(\nu\sqrt{1-\mu^2}\cos\varphi' - \mu\sqrt{1-\nu^2}\cos\varphi) \qquad (5.26)\\
M_{12}(\mu,\varphi,\nu,\varphi') &= -\zeta\mu\Lambda_1\cos\varphi\sin\varphi' - \Lambda_3\mu\cos\varphi'\sin\varphi \\
&- i f_{\mathrm{e}}\zeta\Lambda_2\sqrt{1-\mu^2}\sin\varphi', \qquad (5.27)\\
M_{21}(\mu,\varphi,\nu,\varphi') &= -\Lambda_3\nu\cos\varphi\sin\varphi' - \zeta\Lambda_1\nu\cos\varphi'\sin\varphi + i f_{\mathrm{e}}\zeta\Lambda_2\sqrt{1-\nu^2}, \qquad (5.28)\\
M_{22}(\mu,\varphi,\nu,\varphi') &= \zeta\Lambda_1\sin\varphi'\sin\varphi - \Lambda_3\cos\varphi'\cos\varphi. \qquad (5.29)
\end{aligned}$$

In this case we can already expect, in comparison with the situation $\hat{k} \parallel \hat{e}_3$, that the transport equations differ depending upon the specific vector polarization. The solution of the system can indeed be written as

$$\Delta_{I}^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \sqrt{1-\mu^2}\Big[\cos\varphi\mathcal{M}_{a}(k,\tau) + \sin\varphi\mathcal{M}_{b}(k,\tau)\Big], \tag{5.30}$$

$$\Delta_Q^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \mu\sqrt{1-\mu^2}\biggl[\cos\varphi\mathcal{N}_a(k,\tau)+\sin\varphi\mathcal{N}_b(k,\tau)\biggr], \tag{5.31}$$

$$\Delta_U^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \sqrt{1-\mu^2}\biggl[-\sin\varphi\mathcal{N}_a(k,\tau)+\cos\varphi\mathcal{N}_b(k,\tau)\biggr], \tag{5.32}$$

$$\Delta_V^{(\mathrm{v})}(\varphi,\mu,k,\tau) = \sqrt{1-\mu^2}\sin 2\varphi\mathcal{V}_a(k,\tau)+\mu\mathcal{V}_b(k,\tau). \tag{5.33}$$

The azimuthal factorization of Eqs. (5.30)–(5.33) is not arbitrary and it is dictated by the specific form of the system of Eqs. (5.1)–(5.4) in the case when $\hat{k}\perp\hat{e}_3$. Starting with the polarization W_a the corresponding evolution equations for $\mathcal{M}_a$, $\mathcal{N}_a$ and $\mathcal{V}_a$ are given by:

$$\partial_\tau\mathcal{M}_a + (ik\mu+\epsilon')\mathcal{M}_a + i\mu\partial_\tau W_a - \epsilon' v_a = -\epsilon'\mu\zeta^2(\Lambda_1^2 - f_{\mathrm{e}}^2\Lambda_2^2)\Sigma_{1a}^{(\mathrm{v})}, \tag{5.34}$$

$$\partial_\tau\mathcal{N}_a + (ik\mu+\epsilon')\mathcal{N}_a = -\epsilon'\mu\zeta^2(\Lambda_1^2 - f_{\mathrm{e}}^2\Lambda_2^2)\Sigma_{1a}^{(\mathrm{v})}, \tag{5.35}$$

$$\partial_\tau\mathcal{V}_a + (ik\mu+\epsilon')\mathcal{V}_a = 0, \tag{5.36}$$

where

$$\Sigma_{1a}^{(\mathrm{v})} = \frac{3}{8}\int_{-1}^{1} d\nu[\nu(\nu^2-1)\mathcal{M}_a + (\nu^4-1)\mathcal{N}_a]. \tag{5.37}$$

Consider then the case of the polarization W_b. The evolution equations are, in this second case,

$$\begin{aligned}&\partial_\tau\mathcal{M}_b + (ik\mu+\epsilon')\mathcal{M}_b + i\mu\partial_\tau W_b - \epsilon' v_b = \epsilon'\mu\zeta\Lambda_1\Lambda_3\Sigma_{1b}^{(\mathrm{v})}\\ &-\frac{3}{4}f_{\mathrm{e}}\zeta\Lambda_2\Lambda_3\mu\int_{-1}^{1}\nu^2\mathcal{V}_b,\end{aligned} \tag{5.38}$$

$$\begin{aligned}&\partial_\tau\mathcal{N}_b + (ik\mu+\epsilon')\mathcal{N}_b = \epsilon'\zeta\Lambda_1\Lambda_3\Sigma_{1b}^{(\mathrm{v})}\\ &-\frac{3}{4}f_{\mathrm{e}}\zeta\Lambda_2\Lambda_3\int_{-1}^{1}\nu^2\mathcal{V}_b d\nu,\end{aligned} \tag{5.39}$$

$$\begin{aligned}&\partial_\tau\mathcal{V}_b + (ik\mu+\epsilon')\mathcal{V}_b = \epsilon' f_{\mathrm{e}}\mu\zeta\Lambda_2\Lambda_3\Sigma_{1b}^{(\mathrm{v})}\\ &-\frac{3}{4}f_{\mathrm{e}}\zeta\Lambda_1\Lambda_3\mu\int_{-1}^{1}\nu^2\mathcal{V}_b d\nu,\end{aligned} \tag{5.40}$$

where

$$\Sigma_{1b}^{(\mathrm{v})} = \frac{3}{8}\int_{-1}^{1} d\nu[\nu(\nu^2-1)\mathcal{M}_b + (\nu^4-1)\mathcal{N}_b]. \tag{5.41}$$

The two sets of equations reported in Eqs. (5.34)–(5.36) and in Eqs. (5.37)–(5.40) show various interesting features which can be summarized as follows:

- if $\hat{e}_3 \parallel \hat{k}$ (i.e. $\alpha=\beta=0$) the evolution equations of the two vector polarizations are independent insofar as they can be given different initial conditions prior to decoupling but the evolution equations of the corresponding brightness perturbations are the same;

- if $\hat{e}_3\perp\hat{k}$ (e.g. $\alpha=\beta=-\pi/2$) the two linear polarizations are equally independent but obey different evolution equations as it is clear by comparing Eqs. (5.34)–(5.36) with Eqs. (5.38)–(5.40);

- the equation for the a-vector polarization (i.e. Eqs. (5.34)–(5.36)) lead to linear and circular photon polarizations which are a factor $\mathcal{O}(f_{\rm e})$ smaller than the corresponding equations for the b-vector polarization (see Eqs. (5.37)–(5.40)).

When $\hat{k} \perp \hat{e}_3$ and $\hat{k} = \hat{z}$ we also have that $\hat{a} = \hat{x}$ and $\hat{b} = \hat{y}$. But if $\alpha = \beta = -\pi/2$, then $\hat{e}_3 = \hat{y}$. Therefore the amount of magnetically induced linear and circular photon polarization is larger when the magnetic field and the vector polarization are oriented along the same direction. In Fig. 2 where the geometric set-up of the vector problem

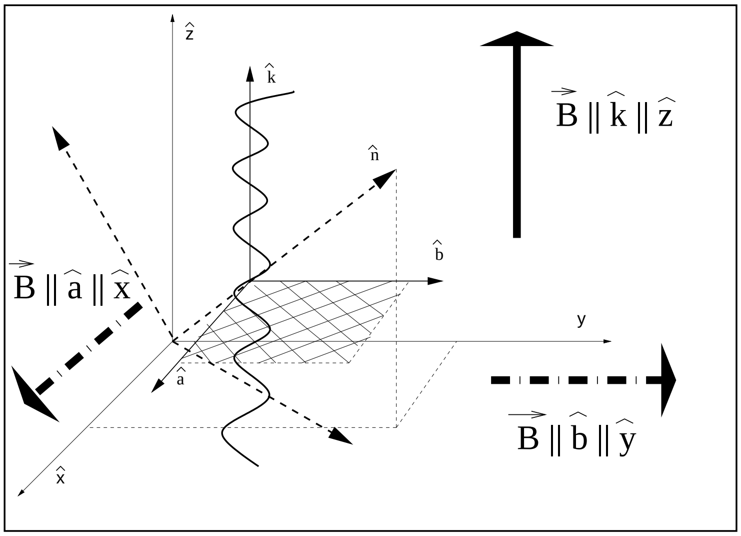

Figure 2: The interplay between the two linear vector polarizations (i.e. $\hat{a}$ and $\hat{b}$), the magnetic field direction and the direction of propagation of the scattered radiation (i.e. $\hat{n}$).

is summarized. The wiggly line represents pictorially a vector wave propagating in the direction $\hat{k}$ which has been taken to be aligned with the $\hat{z}$ axis. Always in Fig. 2 the shaded plane denotes the polarization plane of the vector wave spanned by the two unit vectors $\hat{a}$ and $\hat{b}$. Finally $\hat{n}$ denotes the direction of propagation of the photons. If the direction of the magnetic field is parallel to the direction in which the vector modes propagate (thick arrow in Fig. 2), the photons do not inherit a computable amount of circular polarization and, furthermore, the two linear vector polarizations will lead to the same transport equations for the brightness perturbations. Conversely, if the magnetic field is parallel to one of the two vector polarizations (thick dashed arrows in Fig. 2) the transport equations for the two linear vector polarizations will be different. If the linear vector polarization is aligned with the magnetic field intensity (for instance $\vec{B} \parallel \hat{b}$) the transport equations for $\mathcal{M}_a$, $\mathcal{N}_a$ and $\mathcal{V}_a$ will lead to a V-mode polarization larger than the one generated by the other vector polarization and described in terms of $\mathcal{M}_b$, $\mathcal{N}_b$ and $\mathcal{V}_b$.

The source terms for the evolution equations of the vector modes can be averaged over the orientations of the magnetic field, i.e.

$$\mathcal{L}_I^{(\rm v)}(\mu, \varphi, \vec{k}, \tau) = \epsilon' \hat{n}^i v_i^{(\rm v)}$$

$$+\frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin\beta d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{I}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.42}$$

$$\mathcal{L}_{Q}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin\beta d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{\mathrm{Q}}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.43}$$

$$\mathcal{L}_{U}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin\beta\, d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{\mathrm{U}}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{5.44}$$

$$\mathcal{L}_{V}^{(\mathrm{v})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin\beta d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{\mathrm{V}}^{(\mathrm{v})}(\mu,\nu,\varphi,\varphi',\alpha,\beta). \tag{5.45}$$

The direct computation of the averaged source terms leads to the same expression for both vector polarizations. Denoting with $\mathcal{M}$ either $\mathcal{M}_a$ or $\mathcal{M}_b$ (and similarly for $\mathcal{N}$ and $\mathcal{V}$), Eqs. (5.42), (5.43), (5.44) and (5.45) lead to the following triplet of equations:

$$\begin{aligned}
\partial_\tau \mathcal{M} + (ik\mu + \epsilon')\mathcal{M} + i\mu\partial_\tau W - \epsilon' v &= \frac{\mu}{15}\epsilon'\Big[\Big(5 f_{\mathrm{e}}^2 \Lambda_2^2 - 7\Lambda_1^2\Big) \\
&+ 6\zeta\Lambda_1\Lambda_3 - 2\Lambda_3^2\Big]\Sigma_1^{(\mathrm{v})}, &(5.46)\\
\partial_\tau \mathcal{N} + (ik\mu + \epsilon')\mathcal{N} &= \frac{\mu}{15}\epsilon'\Big[\Big(5 f_{\mathrm{e}}^2 \Lambda_2^2 - 7\Lambda_1^2\Big) \\
&+ 6\zeta\Lambda_1\Lambda_3 - 2\Lambda_3^2\Big]\Sigma_1^{(\mathrm{v})}, &(5.47)\\
\partial_\tau \mathcal{V} + (ik\mu + \epsilon')\mathcal{V} &= \frac{\epsilon'}{3}\zeta[2\Lambda_1\Lambda_3 - \zeta(\Lambda_1^2 + f_{\mathrm{e}}^2\Lambda_2^2)]\Sigma_2^{(\mathrm{v})}. &(5.48)
\end{aligned}$$

As in the case of Eqs. (4.28)–(4.31) if the magnetic field has a predominant direction over typical scales comparable with the wavelengths of the scattered photons, the circular polarization of the photons induced by the vector modes is larger than in the case where, over the same physical scales the magnetic field does not have a specific orientation. This conclusion can be reached by comparing Eqs. (5.46), (5.47) and (5.48) to Eqs. (5.37)–(5.40) obtained in the case when the magnetic field is oriented along one of the two polarizations of the relic vector.

6 Tensor modes

Recalling the notations introduced in Eqs. (3.27)–(3.30), the evolution equations for the tensor components of the brightness perturbations shall be written, in general terms, as

$$\mathcal{L}_{I}^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{I}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{6.1}$$

$$\mathcal{L}_{Q}^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{\mathrm{Q}}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{6.2}$$

$$\mathcal{L}_{U}^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{\mathrm{U}}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{6.3}$$

$$\mathcal{L}_{V}^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_{0}^{2\pi} d\varphi'\, \mathcal{F}_{\mathrm{V}}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta). \tag{6.4}$$

Consider first the case where the propagation of the relic graviton is parallel to the direction of the magnetic field intensity. The direction of propagation of the tensor wave can be chosen, without loss of generality, as $\hat{k} = \hat{z}$. Therefore Eq. (2.5) implies that $\alpha = \beta = 0$, i.e. $\hat{k} \parallel \hat{e}_3$. To illustrate the azimuthal dependence of the problem it is instructive to write down Eq. (6.1) in explicit terms:

$$\partial_\tau \Delta_I^{(\mathrm{t})} + (ik\mu + \epsilon')\Delta_I^{(\mathrm{t})} - \frac{1}{2}(1-\mu^2)\Big[\cos 2\varphi \partial_\tau h_{\oplus} + \sin 2\varphi \partial_\tau h_{\otimes}\Big]$$
$$= \frac{3\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi'\, \mathcal{F}_I^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \tag{6.5}$$

where, according to Eq. (A.7) the integrand of the source term acts on the vector components of the various brightness perturbations and it is given by

$$\mathcal{F}_I^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta) = \mathcal{T}_{II}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_I^{(\mathrm{t})}(\nu,\varphi') + \mathcal{T}_{IQ}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_Q^{(\mathrm{t})}(\nu,\varphi')$$
$$+\mathcal{T}_{IU}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_U^{(\mathrm{t})}(\nu,\varphi') + \mathcal{T}_{IV}(\mu,\nu,\varphi,\varphi',\alpha,\beta)\Delta_V^{(\mathrm{t})}(\nu,\varphi'). \tag{6.6}$$

The remaining three equations (i.e. Eqs. (6.2), (6.3) and (6.4)) have a similar structure but the contribution of the tensor modes of the geometry is absent. The azimuthal dependence can be decoupled from the radial dependence and the brightness perturbations will be

$$\Delta_I^{(\mathrm{t})}(\varphi,\mu,k,\tau) = (1-\mu^2)\Big[\cos 2\varphi \mathcal{Z}_{\oplus}(\mu,k,\tau) + \sin 2\varphi \mathcal{Z}_{\otimes}(\mu,k,\tau)\Big], \tag{6.7}$$

$$\Delta_Q^{(\mathrm{t})}(\varphi,\mu,k,\tau) = (1+\mu^2)\Big[\cos 2\varphi \mathcal{T}_{\oplus}(\mu,k,\tau) + \sin 2\varphi \mathcal{T}_{\otimes}(\mu,k,\tau)\Big], \tag{6.8}$$

$$\Delta_U^{(\mathrm{t})}(\varphi,\mu,k,\tau) = 2\mu\Big[-\sin 2\varphi \mathcal{T}_{\oplus}(\mu,k,\tau) + \cos 2\varphi \mathcal{T}_{\otimes}(\mu,k,\tau)\Big], \tag{6.9}$$

$$\Delta_V^{(\mathrm{t})}(\varphi,\mu,k,\tau) = 2\mu\Big[\cos 2\varphi \mathcal{S}_{\oplus}(\mu,k,\tau) + \sin 2\varphi \mathcal{S}_{\otimes}(\mu,k,\tau)\Big]. \tag{6.10}$$

In the case $\hat{k} \parallel \hat{e}_3$ the symmetry of the system implies necessarily an ansatz in the form of Eqs. (6.7), (6.8), (6.9) and (6.10). Starting with the explicit form of Eq. (6.5), it is immediately clear that the form of $\Delta_I^{(\mathrm{t})}(\varphi,\mu,k,\tau)$ is constrained by the φ dependence appearing at the left hand side of Eq. (6.5). The integrand at the right hand side of Eq. (6.5), i.e. Eq. (6.6), contains also $\Delta_Q^{(\mathrm{t})}(\varphi',\nu,k,\tau)$ whose explicit form is univocally determined by observing that the integral over φ' must match with the φ dependence appearing at the left hand side of Eq. (6.5). But the obtained ansatz for $\Delta_I^{(\mathrm{t})}(\varphi,\mu,k,\tau)$ and $\Delta_Q^{(\mathrm{t})}(\varphi,\mu,k,\tau)$ can be inserted back into Eq. (6.2): this step will constructively determine the explicit form of $\Delta_U^{(\mathrm{t})}(\varphi,\mu,k,\tau)$. Equation (6.3) will finally determine the explicit form of $\Delta_V^{(\mathrm{t})}(\varphi,\mu,k,\tau)$ whose φ dependence will have to be consistent with Eq. (6.4). The result of this procedure, expressed by Eqs. (6.7), (6.8), (6.9) and (6.10), determines the radial evolution and, in particular, the following set of equations [25]:

$$\partial_\tau \mathcal{Z} + (ik\mu + \epsilon')\mathcal{Z} - \frac{1}{2}\partial_\tau h = \epsilon' \zeta^2(\omega)[\Lambda_1^2(\omega) - f_{\mathrm{e}}^2(\omega)\Lambda_2^2(\omega)]\Sigma^{(\mathrm{t})}, \tag{6.11}$$

$$\partial_\tau \mathcal{T} + (ik\mu + \epsilon')\mathcal{T} + \epsilon'\mathcal{T} = -\epsilon'\zeta^2(\omega)[\Lambda_1^2(\omega) - f_{\rm e}^2(\omega)\Lambda_2^2(\omega)]\Sigma^{(\rm t)}, \tag{6.12}$$

$$\partial_\tau \mathcal{S} + (ik\mu + \epsilon')\mathcal{S} = 0, \tag{6.13}$$

where $\mathcal{Z}$, $\mathcal{T}$ and $\mathcal{S}$ denote either the $\oplus$ or the $\otimes$ polarization. By expanding $\mathcal{Z}$, $\mathcal{T}$ in series of Legendre polynomials

$$\mathcal{Z}(\nu, k, \tau) = \sum_\ell (-i)^\ell (2\ell + 1)\, P_\ell(\nu)\, \mathcal{Z}_\ell(k, \tau). \tag{6.14}$$

$$\mathcal{T}(\nu, k, \tau) = \sum_\ell (-i)^\ell (2\ell + 1)\, P_\ell(\nu)\, \mathcal{T}_\ell(k, \tau), \tag{6.15}$$

the source term $\Sigma^{(\rm t)}$ can also be expressed as

$$\begin{aligned} \Sigma^{(\rm t)} &= \frac{3}{32}\int_{-1}^{1} d\nu [(1-\nu^2)^2 \mathcal{Z}(\nu) - (1+\nu^2)^2\mathcal{T}(\nu) - 4\nu^2\mathcal{T}(\nu)] \\ &= \frac{3}{70}\mathcal{Z}_4 + \frac{\mathcal{Z}_2}{7} - \frac{\mathcal{Z}_0}{10} - \frac{3}{70}\mathcal{T}_4 + \frac{6}{7}\mathcal{T}_2 - \frac{3}{5}\mathcal{T}_0, \end{aligned} \tag{6.16}$$

where, as usual, $\mathcal{Z}_\ell$ and $\mathcal{T}_\ell$ denote the ℓ-th mulipoles of the corresponding functions. The results obtained in Eqs. (6.11), (6.12) and (6.13) with the partial treatment of the tensor modes developed in [25]. As in the case of the vectors instead of working with the linear

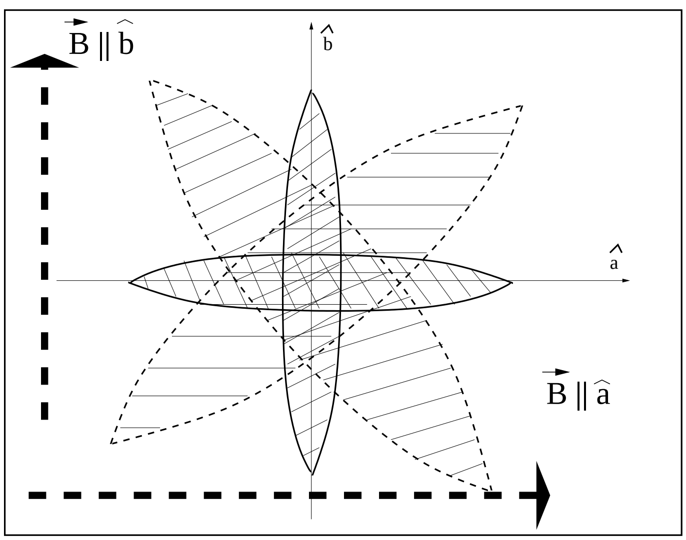

Figure 3: The $\oplus$ and $\otimes$ polarizations are illustrated, respectively, with full and dashed lines. The direction of propagation of the wave is not shown and it is orthogonal to the plane spanned by $\hat{a}$ and $\hat{b}$. When $\vec{B} \parallel \hat{k}$ the magnetic field is oriented perpendicularly to the plane of the figure.

polarizations of the relic gravitons, we could as well work with the circular polarization. The results obtained so far can be easily translated to the case when the relic gravitons are circularly polarized, always assuming that $\hat{k} \parallel \hat{e}_3$, i.e. that the direction of propagation of the relic gravitons is parallel to the orientation of the magnetic field intensity.

As expected from the vector case, when the relic tensor propagates orthogonally to the magnetic field direction, the two tensor polarizations will obey different equations but, at the same time, there will be differences in comparison with the vector case. The two polarizations of the relic gravitons when projected along the directions of the photon propagation will lead to a quadrupole term. The $\oplus$ and $\otimes$ polarization of the tensor mode are illustrated in Fig. 3 which should be compared with the shaded area of Fig. 2. When the $\oplus$ polarization propagates orthogonally to the magnetic field direction its evolution equations are given by:

$$\partial_\tau \mathcal{Z}_\oplus + (ik\mu + \epsilon')\mathcal{Z}_\oplus - \frac{1}{2}\partial_\tau h_\oplus = \frac{\epsilon'}{2}(\zeta^2\Lambda_1^2 + \Lambda_3^2)\Sigma_\oplus^{(\mathrm{t})}$$
$$+\frac{\epsilon'}{2(1-\mu^2)\cos 2\varphi}\Big[\Lambda_3^2(1+\mu^2) - \zeta^2\Big(\Lambda_1^2(1+\mu^2) - 2f_{\mathrm{e}}^2\Lambda_2^2(\mu^2-1)\Big)\Big]\Sigma_\oplus^{(\mathrm{t})}, \quad (6.17)$$
$$\partial_\tau \mathcal{T}_\oplus + (ik\mu + \epsilon')\mathcal{T}_\oplus = -\frac{\epsilon'}{2}(\zeta^2\Lambda_1^2 + \Lambda_3^2)\Sigma_\oplus^{(\mathrm{t})}$$
$$-\frac{\epsilon'}{2(1-\mu^2)\cos 2\varphi}\Big[\Lambda_3^2(1+\mu^2) - \zeta^2\Big(\Lambda_1^2(1+\mu^2) - 2f_{\mathrm{e}}^2\Lambda_2^2(\mu^2-1)\Big)\Big]\Sigma_\oplus^{(\mathrm{t})}, \quad (6.18)$$
$$\partial_\tau \mathcal{S}_\oplus + (ik\mu + \epsilon')\mathcal{S}_\oplus = -\frac{3}{16}f_{\mathrm{e}}\zeta^2\Lambda_1\Lambda_2\frac{\sqrt{1-\mu^2}}{\mu}\frac{\sin\varphi}{\cos 2\varphi}\Sigma_\oplus^{(\mathrm{t})}, \quad (6.19)$$

where it has been assumed that $\alpha = \beta = -\pi/2$. When the $\otimes$ polarization propagates orthogonally to the magnetic field direction its evolution equations are given by:

$$\partial_\tau \mathcal{Z}_\otimes + (ik\mu + \epsilon')\mathcal{Z}_\otimes - \frac{1}{2}\partial_\tau h_\otimes = -\epsilon'\zeta\Lambda_1\Lambda_3\Sigma_\otimes^{(\mathrm{t})}, \quad (6.20)$$
$$\partial_\tau \mathcal{T}_\otimes + (ik\mu + \epsilon')\mathcal{T}_\otimes = -\epsilon'\zeta\Lambda_1\Lambda_3\Sigma_\otimes^{(\mathrm{t})}, \quad (6.21)$$
$$\partial_\tau \mathcal{S}_\otimes + (ik\mu + \epsilon')\mathcal{S}_\otimes = -\epsilon' f_{\mathrm{e}}\zeta\Lambda_1\Lambda_3\frac{\cos\varphi}{\sin 2\varphi}\frac{\sqrt{1-\mu^2}}{\mu}\Sigma_\otimes^{(\mathrm{t})}. \quad (6.22)$$

Equations (6.17), (6.18) and (6.19) can then be compared to Eqs. (6.20), (6.21) and (6.22) recalling that, now the magnetic field is directed along $\hat{y}$. The two tensor polarizations read $\hat{\epsilon}_{ij}^{\oplus} = (\hat{a}_i\hat{a}_j - \hat{b}_i\hat{b}_j)$ and $\hat{\epsilon}_{ij}^{\otimes} = (\hat{a}_i\hat{b}_j + \hat{a}_j\hat{b}_i)$. But since it has been assumed that $\hat{k} = \hat{z}$ we shall also have $\hat{a} = (1,\,0,\,0) = \hat{x}$ and $\hat{b} = (0,\,1,\,0) = \hat{y}$. The addition of a magnetic field either along $\hat{a}$ or along $\hat{b}$ (i.e. orthogonally to $\hat{k}$) is illustrated in Fig. 3 for the two tensor polarizations. The polarization $\oplus$ spans the shaded area bounded by the full lines. The polarization $\otimes$ spans the shaded area bounded by the dashed lines. The effect of having an extra source of circular dichroism along the $\hat{a}$ (or along the $\hat{b}$ axis) will be felt by both tensor polarizations as quantitatively established in Eqs. (6.17)–(6.19) and in Eqs. (6.20)–(6.22).

The last step is to compute the evolution equations by averaging the source functions over the directions of the magnetic field.

$$\mathcal{L}_I^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1}d\nu\int_0^{2\pi}d\varphi'\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi}d\alpha\mathcal{F}_I^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), (6.23)$$
$$\mathcal{L}_Q^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1}d\nu\int_0^{2\pi}d\varphi'\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi}d\alpha\mathcal{F}_{\mathrm{Q}}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), (6.24)$$

$$\mathcal{L}_U^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\alpha \mathcal{F}_{\mathrm{U}}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta), \quad (6.25)$$

$$\mathcal{L}_V^{(\mathrm{t})}(\mu,\varphi,\vec{k},\tau) = \frac{3\epsilon'}{128\pi^2}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\alpha \mathcal{F}_{\mathrm{V}}^{(\mathrm{t})}(\mu,\nu,\varphi,\varphi',\alpha,\beta). \quad (6.26)$$

The result for the evolution equations of the tensor polarizations with averaged sources is given by:

$$\partial_\tau \mathcal{Z} + (ik\mu + \epsilon')\mathcal{Z} - \frac{1}{2}\partial_\tau h = \frac{\epsilon'}{15}[\zeta^2(7\Lambda_1^2 - 5f_{\mathrm{e}}^2\Lambda_2^2) - 6\zeta\Lambda_1\Lambda_3 + 2\Lambda_3^2]\Sigma^{(\mathrm{t})}, \quad (6.27)$$

$$\partial_\tau \mathcal{T} + (ik\mu + \epsilon')\mathcal{T} = -\frac{\epsilon'}{15}[\zeta^2(7\Lambda_1^2 - 5f_{\mathrm{e}}^2\Lambda_2^2) - 6\zeta\Lambda_1\Lambda_3 + 2\Lambda_3^2]\Sigma^{(\mathrm{t})}, \quad (6.28)$$

$$\partial_\tau \mathcal{S} + (ik\mu + \epsilon')\mathcal{S} = 0 \quad (6.29)$$

These results extend and partially correct the results derived in [25]. The correction has to do with the source term of Eq. (6.29) which vanishes exactly unlike stated in [25] because of an error in the azimuthal integrations.

7 Concluding remarks

An arbitrarily oriented magnetic field has been incorporated in the Stokes matrix of the last electron-photon scattering. The transport equations for the scalar, vector and tensor components of the brightness perturbations have been derived and studied in various physical situations. The obtained results pave the way for a consistent improvement of the available analytical and numerical tools used for the calculation of magnetized CMB anisotropies. The general treatment developed here is also expected to be relevant for the careful assessment of the level of circular polarization induced at last scattering by the presence of a small-scale component of the pre-decoupling magnetic field.

A Stokes and Mueller matrices

The explicit form of the Stokes and Mueller matrices for arbitrary orientation of the magnetic field will now be reported. The four distinct entries of the Stokes matrix $M(\Omega, \Omega', \alpha, \beta)$ appearing in Eq. (2.14) can be written, in explicit terms, as

$$
\begin{aligned}
& M_{11}(\Omega, \Omega', \alpha, \beta) = \frac{\zeta\Lambda_1 - \Lambda_3}{2}\Big[\sqrt{1-\mu^2}\sqrt{1-\nu^2} + \mu\nu\cos(\varphi-\alpha)\cos(\varphi'-\alpha)\Big] \\
& + \frac{\zeta\Lambda_1 + \Lambda_3}{2}\Big\{\cos 2\beta\Big[\mu\nu\cos(\varphi-\alpha)\cos(\varphi'-\alpha) - \sqrt{1-\mu^2}\sqrt{1-\nu^2}\Big] \\
& + \sin 2\beta\Big[\mu\sqrt{1-\nu^2}\cos(\varphi-\alpha) + \nu\sqrt{1-\mu^2}\cos(\varphi'-\alpha)\Big]\Big\} \\
& + \zeta\Lambda_1\mu\nu\sin(\varphi-\alpha)\sin(\varphi'-\alpha) + i f_{\rm e}\zeta\Lambda_2\Big\{\sin\beta\Big[\mu\sqrt{1-\nu^2}\sin(\varphi-\alpha) \\
& - \nu\sqrt{1-\mu^2}\sin(\varphi'-\alpha)\Big] + \mu\nu\cos\beta\sin(\varphi-\varphi')\Big\}, \qquad \text{(A.1)} \\
& M_{12}(\Omega, \Omega', \alpha, \beta) = \frac{\Lambda_3 - \Lambda_1\zeta}{2}\mu\sin(\varphi'-\alpha)\cos(\varphi-\alpha) \\
& - \frac{\Lambda_3 + \Lambda_1\zeta}{2}\Big\{\mu\sin(\varphi'-\alpha)\cos(\varphi-\alpha)\cos 2\beta + \sqrt{1-\mu^2}\sin(\varphi'-\alpha)\sin 2\beta\Big\} \\
& + \zeta\Lambda_1\mu\sin(\varphi-\alpha)\cos(\varphi'-\alpha) \\
& - i f_{\rm e}\zeta\Lambda_2\Big[\mu\cos\beta\cos(\varphi'-\varphi) + \sqrt{1-\mu^2}\sin\beta\cos(\varphi'-\alpha)\Big], \qquad \text{(A.2)} \\
& M_{21}(\Omega, \Omega', \alpha, \beta) = -\frac{\zeta\Lambda_1 + \Lambda_3}{2}\sqrt{1-\nu^2}\ \sin 2\beta\sin(\varphi-\alpha) \\
& + \frac{\nu}{4}(\zeta\Lambda_1 + \Lambda_3)\Big[\sin(\varphi+\varphi'-2\alpha) - \sin(\varphi'-\varphi)\Big] \\
& - \frac{\nu}{4}(\zeta\Lambda_1 + \Lambda_3)\Big[\sin(\varphi+\varphi'-2\alpha) - \sin(\varphi'-\varphi)\Big]\cos 2\beta \\
& i f_{\rm e}\Lambda_2\zeta\Big[\nu\cos\beta\cos(\varphi'-\varphi) + \sqrt{1-\nu^2}\sin\beta\cos(\varphi-\alpha)\Big] \qquad \text{(A.3)} \\
& M_{22}(\Omega, \Omega', \alpha, \beta) = \frac{\zeta\Lambda_1 + \Lambda_3}{4}\sin 2\alpha(1-\cos 2\beta)\sin(\varphi'+\varphi) - \Lambda_3\cos(\varphi'-\varphi) \\
& + \frac{\zeta\Lambda_1 + \Lambda_3}{4}(1+\cos 2\beta)[\cos(\varphi'-\varphi) - \cos 2\alpha\cos(\varphi'+\varphi)] \\
& + \frac{\zeta\Lambda_1 + \Lambda_3}{2}\Big[\cos(\varphi'-\varphi) + \cos 2\alpha\cos(\varphi'+\varphi)\Big] \\
& - i f_{\rm e}\zeta\Lambda_2\cos\beta\sin(\varphi'-\varphi). \qquad \text{(A.4)}
\end{aligned}
$$

Equations (A.1), (A.2), (A.3) and (A.4) have various interesting limits which depend upon the specific orientation of the magnetic field intensity. In the absence of magnetic field we have that

$$\Lambda_1 \to 1, \qquad \Lambda_2 \to 1, \qquad \Lambda_3 \to 1, \qquad \zeta \to -1, \qquad f_{\rm e} \to 0. \tag{A.5}$$

In the limit defined by Eq. (A.5), Eqs. (A.1)–(A.4) reduce to

$$\begin{aligned}
&M_{11}(\mu,\varphi,\nu,\varphi') = -\sqrt{1-\mu^2}\sqrt{1-\nu^2} - \mu\nu\cos(\varphi'-\varphi),\\
&M_{12}(\mu,\varphi,\nu,\varphi') = \mu\sin(\varphi'-\varphi), \qquad M_{21}(\mu,\varphi,\nu,\varphi') = -\nu\sin(\varphi'-\varphi),\\
&M_{22}(\mu,\varphi,\nu,\varphi') = -\cos(\varphi'-\varphi),
\end{aligned} \tag{A.6}$$

where, as already pointed out in section 2, $\mu = \cos\vartheta$ and $\nu = \cos\vartheta'$. The integrands appearing in the source terms of Eqs. (2.17), (2.18), (2.19 and (2.20) are given by

$$\begin{aligned}
&\mathcal{F}_I(\Omega,\Omega',\alpha,\beta) = \mathcal{T}_{II}(\Omega,\Omega',\alpha,\beta) + \mathcal{T}_{IQ}(\Omega,\Omega',\alpha,\beta)Q(\Omega')\\
&+\mathcal{T}_{IU}(\Omega,\Omega',\alpha,\beta)U(\Omega') + \mathcal{T}_{IV}(\Omega,\Omega',\alpha,\beta)V(\Omega'), \qquad (A.7)\\
&\mathcal{F}_Q(\Omega,\Omega',\alpha,\beta) = \mathcal{T}_{QI}(\Omega,\Omega',\alpha,\beta)I(\Omega') + \mathcal{T}_{QQ}(\Omega,\Omega',\alpha,\beta)Q(\Omega')\\
&+\mathcal{T}_{QU}(\Omega,\Omega',\alpha,\beta)U(\Omega') + \mathcal{T}_{QV}(\Omega,\Omega',\alpha,\beta)V(\Omega'), \qquad (A.8)\\
&\mathcal{F}_U(\Omega,\Omega',\alpha,\beta) = \mathcal{T}_{UI}(\Omega,\Omega',\alpha,\beta)I(\Omega') + \mathcal{T}_{UQ}(\Omega,\Omega',\alpha,\beta)Q(\Omega')\\
&+\mathcal{T}_{UU}(\Omega,\Omega',\alpha,\beta)U(\Omega') + \mathcal{T}_{UV}(\Omega,\Omega',\alpha,\beta)V(\Omega'), \qquad (A.9)\\
&\mathcal{F}_V(\Omega,\Omega',\alpha,\beta) = \mathcal{T}_{VI}(\Omega,\Omega',\alpha,\beta)I(\Omega') + \mathcal{T}_{VQ}(\Omega,\Omega',\alpha,\beta)Q(\Omega')\\
&+\mathcal{T}_{VU}(\Omega,\Omega',\alpha,\beta)U(\Omega') + \mathcal{T}_{VV}(\Omega,\Omega',\alpha,\beta)V(\Omega'), \qquad (A.10)
\end{aligned}$$

where the matrix elements $\mathcal{T}_{ij}$ are computed in terms of Eqs. (A.1), (A.2), (A.3) and (A.4):

$$\mathcal{T}_{II} = 2[|M_{11}|^2 + |M_{12}|^2 + |M_{21}|^2 + |M_{22}|^2], \tag{A.11}$$
$$\mathcal{T}_{IQ} = 2[|M_{11}|^2 - |M_{12}|^2 + |M_{21}|^2 - |M_{22}|^2], \tag{A.12}$$
$$\mathcal{T}_{IU} = 2[M_{11}M_{12}^* + M_{12}M_{11}^* + M_{21}M_{22}^* + M_{22}M_{21}^*], \tag{A.13}$$
$$\mathcal{T}_{IV} = 2[i(M_{12}M_{11}^* - M_{12}^*M_{11} + M_{22}M_{21}^* - M_{21}M_{22}^*], \tag{A.14}$$
$$\mathcal{T}_{QI} = 2[|M_{11}|^2 + |M_{12}|^2 - |M_{21}|^2 - |M_{22}|^2], \tag{A.15}$$
$$\mathcal{T}_{QQ} = 2[|M_{11}|^2 - |M_{12}|^2 - |M_{21}|^2 + |M_{22}|^2], \tag{A.16}$$
$$\mathcal{T}_{QU} = 2[M_{11}M_{12}^* + M_{12}M_{11}^* - M_{21}M_{22}^* - M_{22}M_{21}^*], \tag{A.17}$$
$$\mathcal{T}_{QV} = 2[i(M_{12}M_{11}^* - M_{12}^*M_{11} - M_{22}M_{21}^* + M_{21}M_{22}^*], \tag{A.18}$$
$$\mathcal{T}_{UI} = 2[M_{11}M_{21}^* + M_{12}M_{22}^* + M_{21}M_{11}^* + M_{22}M_{12}^*], \tag{A.19}$$
$$\mathcal{T}_{UQ} = 2[M_{11}M_{21}^* - M_{12}M_{22}^* + M_{21}M_{11}^* - M_{22}M_{12}^*], \tag{A.20}$$
$$\mathcal{T}_{UU} = 2[M_{11}M_{22}^* + M_{12}M_{21}^* + M_{21}M_{12}^* + M_{22}M_{11}^*], \tag{A.21}$$
$$\mathcal{T}_{UV} = 2[i(M_{12}M_{21}^* - M_{22}^*M_{11} + M_{22}M_{11}^* - M_{12}^*M_{21}], \tag{A.22}$$
$$\mathcal{T}_{VI} = 2[i(M_{11}M_{21}^* + M_{12}M_{22}^* - M_{21}M_{11}^* - M_{22}M_{12}^*], \tag{A.23}$$
$$\mathcal{T}_{VQ} = 2[i(M_{11}M_{21}^* - M_{12}M_{22}^* - M_{21}M_{11}^* + M_{22}M_{12}^*], \tag{A.24}$$
$$\mathcal{T}_{VU} = 2[i(M_{11}M_{22}^* + M_{12}M_{21}^* - M_{21}M_{12}^* - M_{22}M_{11}^*], \tag{A.25}$$
$$\mathcal{T}_{VV} = 2[M_{22}^*M_{11} - M_{12}M_{21}^* + M_{22}M_{11}^* - M_{12}^*M_{21}]. \tag{A.26}$$

In Eqs. (A.11)–(A.26) the explicit dependence upon the six angles has been suppressed only for sake of simplicity. The components of the incident electric fields in the local frame $\hat{e}_1$, $\hat{e}_2$

and $\hat{e}_3$ can be related to the components off the electric field in the three Cartesian directions as

$$\begin{aligned} E_1 &= \cos\alpha\cos\beta E'_x + \sin\alpha\cos\beta E'_y - \sin\beta E'_z, \\ E_2 &= -\sin\alpha E'_x + \cos\alpha E'_y, \\ E_3 &= \cos\alpha\sin\beta E'_x + \sin\alpha\sin\beta E'_y + \cos\beta E'_z. \end{aligned} \tag{A.27}$$

The incident electric fields E'_x, E'_y and E'_z can be related, in turn, to their polar components as:

$$\begin{aligned} E'_x &= \cos\vartheta'\cos\varphi' E'_\vartheta - \sin\varphi' E'_\varphi, \\ E'_y &= \cos\vartheta'\sin\varphi' E'_\vartheta + \cos\varphi' E'_\varphi, \\ E'_z &= -\sin\vartheta' E'_\vartheta, \end{aligned} \tag{A.28}$$

where, as already spelled out in section 2 the direction of propagation of the incident radiation $\hat{n}'$ coincides with $\hat{r}'$ and $(E'_\vartheta,\ E'_\varphi)$ are the components of the incident electric field in the spherical basis. The relations between the outgoing and the ingoing electric fields is given by

$$\begin{aligned} E_\vartheta(\vartheta,\varphi,\vartheta',\varphi',\alpha,\beta) &= \frac{r_{\mathrm{e}}}{r}\Big[M_{11}E_\vartheta(\vartheta',\varphi') + M_{12}E_\vartheta(\vartheta',\varphi')\Big], \\ E_\vartheta(\vartheta,\varphi,\vartheta',\varphi',\alpha,\beta) &= \frac{r_{\mathrm{e}}}{r}\Big[M_{21}E_\vartheta(\vartheta',\varphi') + M_{22}E_\vartheta(\vartheta',\varphi')\Big], \end{aligned} \tag{A.29}$$

where, $M_{ij} \equiv M_{ij}(\vartheta,\varphi,\vartheta',\varphi',\alpha,\beta)$ are given by Eqs. (A.1)–(A.3).

B Angular integrations

The angular integrations of the collision term can always be performed either directly or with the help of the appropriate Rayleigh expansion. The results are however rather lengthy. Furthermore, in the case of the vector and of the tensor modes (i.e. sections 5 and 6), the integrations over φ' change depending upon the the polarization of the vector (or of the tensor) wave. In the scalar case the results of the angular integration over φ' are given hereunder following the notation of Eq. (4.6):

$$\begin{aligned} \overline{\mathcal{T}}_{II} &= 2\pi\nu^2\Big\{(3\mu^2-1) + f_{\mathrm{e}}^2\Big[(1+\mu^2) - 2(1+\mu^2)\sin^2\beta\cos^2(\varphi-\alpha) \\ &+ \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin 2\beta\Big]\Big\} + 2\pi\Big\{3-\mu^2 + f_{\mathrm{e}}^2\Big[(1+\mu^2) \\ &- 2\sin^2\beta(1-\mu^2)\cos^2(\varphi-\alpha) + \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin 2\beta\Big]\Big\} \end{aligned} \tag{B.1}$$

$$\overline{\mathcal{T}}_{IQ} = \frac{\pi}{2}(\nu^2-1)\Big\{4(3\mu^2-1) + f_{\mathrm{e}}^2\Big[4(1-\mu^2)\sin^2(\varphi-\alpha)$$

$$
\begin{aligned}
&+ && (1-\mu^2)\Big(\cos 2(\varphi-\alpha-\beta)+\cos 2(\varphi-\alpha+\beta)\Big)+2\cos 2\beta(1+3\mu^2) \\
&- && 2\mu\sqrt{1-\mu^2}\Big(\sin(\varphi-\alpha-2\beta)-\sin(\varphi-\alpha-2\beta)\Big)\Big], && \text{(B.2)} \\
\overline{\mathcal{T}}_{IU} &= && 0, && \text{(B.3)} \\
\overline{\mathcal{T}}_{IV} &= && 8\pi f_{\mathrm{e}}\nu[(1+\mu^2)\cos\beta+\mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta], && \text{(B.4)} \\
\overline{\mathcal{T}}_{QI} &= && 2\pi\nu^2\Big\{3(\mu^2-1)+f_{\mathrm{e}}^2\Big[(\mu^2-1)+2\sin^2\beta(1+(1-\mu^2)\cos^2(\varphi-\alpha)) \\
&+ && \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin 2\beta\Big]\Big\}+2\pi\Big\{(1-\mu^2) \\
&+ && f_{\mathrm{e}}^2\Big[(\mu^2-1)+2\sin^2\beta\Big[\sin^2(\varphi-\alpha)-\mu^2\cos^2(\varphi-\alpha)\Big] \\
&+ && \mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin 2\beta\Big\} && \text{(B.5)} \\
\overline{\mathcal{T}}_{QQ} &= && \frac{\pi}{2}(1-\nu^2)\Big\{12(1-\mu^2)+f_{\mathrm{e}}^2\Big[2(\mu^2-1) \\
&- && 2(1+\mu^2)\cos 2(\varphi-\alpha)+(1+\mu^2)(\cos 2(\varphi-\alpha-\beta)+\cos 2(\varphi-\alpha+\beta)) \\
&+ && 6\cos 2\beta(1-\mu^2)+2\mu\sqrt{1-\mu^2}(\sin(\varphi-\alpha-2\beta)-\sin(\varphi-\alpha+2\beta))\Big]\Big\}. && \text{(B.6)} \\
\overline{\mathcal{T}}_{QU} &= && 0, && \text{(B.7)} \\
\overline{\mathcal{T}}_{QV} &= && 8\pi f_{\mathrm{e}}\nu[(\mu^2-1)\cos\beta+\mu\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta], && \text{(B.8)} \\
\overline{\mathcal{T}}_{UI} &= && -2\pi f_{\mathrm{e}}^2\sin(\varphi-\alpha)\Big\{\nu^2\Big[4\mu\cos(\varphi-\alpha)\sin^2\beta+\sqrt{1-\mu^2}\sin 2\beta\Big] \\
&+ && \Big[\sqrt{1-\mu^2}\sin 2\beta-4\mu\cos(\varphi-\alpha)\sin^2\beta\Big]\Big\} && \text{(B.9)} \\
\overline{\mathcal{T}}_{UQ} &= && -4\pi f_{\mathrm{e}}^2(\nu^2-1)\sin\beta\sin(\varphi-\alpha)\Big[\sqrt{1-\mu^2}\cos\beta+2\mu\cos(\varphi-\alpha)\sin\beta\Big], && \text{(B.10)} \\
\overline{\mathcal{T}}_{UU} &= && 0, && \text{(B.11)} \\
\overline{\mathcal{T}}_{UV} &= && -8\pi\nu f_{\mathrm{e}}\sqrt{1-\mu^2}\sin(\varphi-\alpha)\sin\beta && \text{(B.12)} \\
\overline{\mathcal{T}}_{VI} &= && 4\pi f_{\mathrm{e}}\Big\{\nu^2\Big[2\mu\cos\beta-\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\Big] \\
&+ && \Big[2\mu\cos\beta+3\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\Big]\Big\}, && \text{(B.13)} \\
\overline{\mathcal{T}}_{VQ} &= && 2\pi f_{\mathrm{e}}(\nu^2-1)\Big[4\mu\cos\beta-2\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\Big], && \text{(B.14)} \\
\overline{\mathcal{T}}_{VU} &= && 0, && \text{(B.15)} \\
\overline{\mathcal{T}}_{VV} &= && 8\pi\nu\Big\{\mu+\cos\beta f_{\mathrm{e}}^2\Big[\mu\cos\beta+\sqrt{1-\mu^2}\cos(\varphi-\alpha)\sin\beta\Big]\Big\}. && \text{(B.16)}
\end{aligned}
$$

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